On the Non-existence of Orthogonal Latin Squares of Order Six

by

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1. Introduction and Definitions

It has been known since the turn of the century that no pair of orthogonal Latin squares exists. Lamy [1900-1917] and Fisher and Yates [1938] proved the non-existence by exhaustive enumeration of all possible Latin squares of order six. The purpose of this paper is to present an analytic proof of the non-existence of such a pair of Latin squares. We note in passing that Hedayat [1969] explicitly formulated the problem and suggests an approach for studying the problem. We utilize a different approach here. Two preliminary theorems, one demonstrating the non-existence of a pair of \( F(6; 33) \) squares and the second showing the unique series of the Biometrika Unit, Cornell University...
how to decompose a Latin square of order six into two orthogonal \( F(6; 2, 3, 2) \) and \( F(6; 3, 3) \) squares, are presented first. These results are used to prove the nonexistence of a pair of orthogonal Latin squares of order six in the third theorem.

An \( F \)-square has been defined as follows by Hedayat \[1969J \] and Hedayat and Seiden \[1970J \]:

Definition 4.1. Let \( A = [a_{ij}] \) be an \( n \times n \) matrix and let \( \Sigma = \{c_1, c_2, \ldots, c_m\} \) be the ordered set of \( m \) distinct elements or symbols of \( A \). In addition, suppose that for each \( i = 1, 2, \ldots, m \), \( c_k \) appears exactly \( \lambda_k \) times \( (\lambda_k \geq 1) \) in each row and column of \( A \). Then \( A \) will be called a frequency square, more concisely, an \( F \)-square on \( \Sigma \) of order \( n \) and frequency vector \( (\lambda_1, \lambda_2, \ldots, \lambda_m) \) and will be denoted by \( F(n; \lambda_1, \lambda_2, \ldots, \lambda_m) \). Note that \( \lambda_1 + \lambda_2 + \cdots + \lambda_m = n \) and that when \( \lambda_k = 1 \) for all \( k \) and \( m = n \), a Latin square, \( LS(n) \), of order \( n \) results.

As with Latin squares, one may consider orthogonality of a pair of \( F \)-squares of the same order. The
above authors have given the following definition to cover this situation:

**Definition 1.2.** Given an $F_1 (r_1, r_2, \ldots, r_k)$ on a set $\Sigma = (a_1, a_2, \ldots, a_k)$ and an $F_2 (n_1, u_1, u_2, \ldots, u_k)$ on a set $\Omega = (b_1, b_2, \ldots, b_k)$, we say $F_2$ is an orthogonal mate for $F_1$ (and write $F_2 \perp F_1$), if upon superposition of $F_2$ on $F_1$, $a_i$ appears $n_i u_j$ times with $b_j$. 
2. Two Preliminary Theorems

Theorem 2.1. An $F(6; 3,3)$-square has no orthogonal mate which is an $F(6; 3,3)$-square.

Proof: First consider the following $F(6; 3,3)$-square for ease of comprehension:

$$
\begin{bmatrix}
A_1 & B_1 \\
B_2 & A_2
\end{bmatrix} = \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix} \times \begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1
\end{pmatrix} =
\begin{bmatrix}
1 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 1
\end{bmatrix}
$$

where $\otimes$ denotes Kronecker product. For $A_i$, insert ones and $9-n$ zeros in any manner desired. Call the $3 \times 3$ matrices $C$. Then in order to have exactly three ones and three zeros in each row and column, we must insert the complement (ones replaced by zeros and vice versa replaced by ones) of $C$, say $D$, in the upper right-hand corner and in the lower left-hand corner to obtain the $6 \times 6$ square.

\[
\begin{bmatrix}
C & D \\
D & X
\end{bmatrix}
\]
where $X$ is the remaining $3 \times 3$ matrix required to produce an $F(6; 3, 3)$-square. In order for $X$ to satisfy the requirement that nine ones and nine zeros occur with the 18 ones in the original square, it is necessary that $X = D$. If $X = D$ then the requirement that ones and zeros in each row and in each column exactly three times is violated. Hence, no orthogonal mate for the original square $\begin{bmatrix} A_1 & B_1 \\ B_2 & A_2 \end{bmatrix}$ can be found.

Note that the proof does not depend upon where the 18 ones are located in the Latin square of order 6, and therefore does not depend upon which $F(6; 3, 3)$-square one uses for the original $F$-square.

The theorem cannot be extended to any other integers since it is a simple matter to construct an orthogonal pair of $F(n, \text{even}; W_2, W_2)$-squares from an $OL(n, 2)$-set. Simply replace the even numbers in each square with $A_1$ and the odd numbers with $A_2$ to form the two $F(n; W_2, W_2)$-squares. If four $F(10; 3, 3)$-squares could be found it would be possible to construct an $OL(10, 4)$-set.
Theorem 2.2. Any Latin square of order six may be decomposed into an \( F(6; 3, 3) \)-square and an \( F(6; 3, 3, 2) \)-square, which are orthogonal, or into an \( F(6; 3, 3) \)-square, an \( F(6; 3, 3, 2) \)-square, and a remainder, which are orthogonal.

Proof: Set up the following four orthogonal contrasts among the six symbols \( (1, 2, 3, 4, 5, 6) \) in any Latin square:

<table>
<thead>
<tr>
<th>orthogonal contrast</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C_1 )</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>( C_2 )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-2</td>
</tr>
<tr>
<td>( C_3 )</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( C_4 )</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>-2</td>
<td>0</td>
</tr>
<tr>
<td>( C_5 )</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

In the given Latin square of order six, replace the odd numbered symbols with a zero and the even numbered symbols with ones to form an \( F(6; 3, 3) \)-square. To form an \( F(6; 3, 3, 2) \)-square simply use contrasts \( C_2 \) and \( C_3 \) which indicate that 1 and 2 may be replaced with zeros, 3 and 4 with one, and 5 and 6 with a two. This square is obviously orthogonal to the \( F(6; 3, 3) \)-square.
In an alternate procedure it is demonstrated that any Latin square of order six decomposes into an \( F(6;3,3) \)-square and an \( F(6;2,2,2) \)-square as to renumber the symbols 1, 2, 3, 4, 5, 6 as 00, 01, 02, 10, 11, 12, respectively. Hence for combination \( i,j \), \( i=0,1 \) and \( j=0,1,2 \), form the \( F(6;3,3) \)-square from the \( i \)th level and form the \( F(6;2,2,2) \)-square from the \( j \)th level of the \( 2 \times 3 \) factorial. The main effects and interaction effects are orthogonal. Thus, the F-squares produce an orthogonal. The interaction forms a remainder whereas the main effects form F-squares.

To form the pair \( F(6;3,3) \)-square and \( F(6;2,2,2) \)-square, simply let the odd numbers be one symbol and the even a second to form the final square, \( F(6;3,3) \)-square.

To form the \( F(6;1,1,1,2) \)-square let any pair of symbols in the Latin square of order six be replaced by a single symbol. It can be shown that it is impossible for a single symbol in this square to appear with each symbol of an \( F(6;3,3) \)-square an equal number of times. Hence, the two squares cannot be orthogonal.
3. The Main Theorem

Theorem 3.1. Any Latin square of order six is orthogonally матлес.

Proof: The proof will be by contradiction. First assume that an \( O_{1}(6;2) \)-set exists, that is \( L_{1}(6) \) where \( L_{1}(6) \) is the six Latin square of order six, \( L_{2}(6) \). From Theorem 2.2 we know that \( L_{1}(6) \) may be decomposed into an \( F(6;3,3) \)-square and an \( F_{1}(6;3,1,1,2) \)-square, which are not orthogonal, or into an \( F_{1}(6;3,3) \)-square an \( F(6;2,3,2) \)-square, and a remainder. The same is true for \( L_{2}(6) \). Then from the assumption

\[
L_{1}(6) \perp L_{2}(6) \Rightarrow (\text{Remainder} \perp F_{1}(6;3,3,2) \perp F_{1}(6;3,3)) \perp \]

\[
(F_{2}(6;3,3) \perp F_{2}(6;2,3,2) \perp \text{Remainder}),
\]

where \( \perp \) means pairwise orthogonal. However we know from Theorem 2.1 that any \( F(6;3,3) \)-square is orthogonally матлес. Hence, the assumption is violated and the proof is completed.