

Complete Sets of Orthogonal F-Squares of Prime  
Power Order with Differing Numbers of Symbols

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Abstract

This paper represents a contribution to the statistical and mathematical theory of orthogonality of latin squares and of F-squares. Orthogonal F-squares are useful in designing experiments wherein the number of treatments is less than the number of rows and columns and wherein several sets of treatments are applied either simultaneously or sequentially on the same set of experimental units. Hedayat, Raghavarao, and Seiden (1975) showed how to construct complete sets of orthogonal F-squares of order  $n = S^m$  where  $S$  is a prime number,  $m$  is a positive integer, and the number of symbols in each square is the same constant number. We show how to construct complete sets of orthogonal F-squares of order  $n = S^m$ , where the F-squares in the sets have differing numbers of symbols. We demonstrate also the relationship between orthogonal latin squares and orthogonal F-squares, in particular we show how to decompose complete sets of orthogonal latin squares into complete sets of orthogonal F-squares.

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## 1. Introduction and Definitions

An F-square has been defined by Hedayat (1969) and Hedayat and Seiden (1970) as follows:

Definition 1.1. Let  $A = [Q_{ij}]$  be an  $n \times n$  matrix and let  $\Sigma = \{A_1, A_2, \dots, A_m\}$  be the ordered set of  $m$  distinct elements or symbols of  $A$ . In addition, suppose that for each  $k = 1, 2, \dots, m$ ,  $A_k$  appears exactly  $\lambda_k$  times ( $\lambda_k \geq 1$ ) in each row and column of  $A$ . Then  $A$  will be called a frequency square or, more concisely, an F-square, on  $\Sigma$  of order  $n$  and frequency vector  $(\lambda_1, \lambda_2, \dots, \lambda_m)$ . The notation we use to denote this F-square is  $F(n; \lambda_1, \lambda_2, \dots, \lambda_m)$ . Note that  $\lambda_1 + \lambda_2 + \dots + \lambda_m = n$  and that when  $\lambda_k = 1$  for all  $k$  and  $m = n$ , a latin square of order  $n$  results.

As with latin squares, one may consider orthogonality of a pair of F-squares of the same order. The above cited authors have given the following definition to cover this situation:

Definition 1.2. Given an F-square  $F_1(n; \lambda_1, \lambda_2, \dots, \lambda_k)$  and an F-square  $F_2(n; u_1, u_2, \dots, u_t)$ , we say  $F_2$  is an orthogonal mate for  $F_1$  (and write  $F_2 \perp F_1$ ), if upon superposition of  $F_2$  on  $F_1$ ,  $A_i$  appears  $\lambda_i u_j$  times with  $B_j$ .

Note that when  $\lambda_i = 1 = u_j$  for all  $i$  and  $j$  and

$$\sum_{i=1}^k \lambda_i = n = \sum_{j=1}^t u_j, \text{ we have the familiar definition of the}$$

orthogonality of two latin squares of order  $n$ .

The definition of a set of orthogonal F-squares is given as:

Definition 1.3. Let  $\{F_1, F_2, \dots, F_t\}$  be a set of two or more F-squares of order  $n \geq 3$ . The F-squares in this set are called orthogonal, and we refer to  $\{F_1, F_2, \dots, F_t\}$  as an orthogonal set, provided that  $F_i$  and  $F_j$  are orthogonal for each  $i \neq j$ . If  $F_1, F_2, \dots, F_t$  are all latin squares of order  $n$  then a set of  $t$  mutually orthogonal latin squares results and is denoted as  $OL(n, t)$ .

If a complete set of orthogonal latin squares of order  $n$  exists, then  $t = n - 1$  and the set is denoted as  $OL(n, n-1)$ . If a complete set of orthogonal F-squares of order  $n$  exists, the number will depend upon the number of symbols in each F-square. This leads to the following definition which is a generalization of the one given by Hedayat and Seiden (1970):

Definition 1.4. A complete set of  $t$  orthogonal F-squares of

order  $n$  is denoted as  $CSOFS(n, t)$ , where  $t = \sum_{i=2}^n N_i$ ,  $N_i$  is

the number of  $F(n; \lambda_1, \lambda_2, \dots, \lambda_i)$ -squares in the set

(i.e.,  $N_i$  is the number of squares with  $i$  distinct

elements),  $\sum_{h=1}^i \lambda_h = n$ , and  $\sum_{i=2}^n N_i (i-1) = (n-1)^2$ .

The fact that  $\sum_{i=2}^n N_i (i-1) = (n-1)^2$  in order to have a

CSOFS follows directly from analysis of variance theory and from factorial theory in that the interaction of two  $n$ -level factors has  $(n-1)^2$  degrees of freedom and from the fact that only interaction degrees of freedom are available to construct  $F$ -squares. For each  $F(n; \lambda_1, \lambda_2, \dots, \lambda_i)$ -square, there are  $(i-1)$  degrees of freedom associated with the  $i$  distinct symbols of an  $F$ -square, there are  $N_i$   $F$ -squares containing  $i$  symbols, and hence  $(n-1)^2 = \sum_{i=2}^n N_i (i-1)$ .

## 2. One-to-One Correspondence Between Factorial Effects and Orthogonal Latin Squares and Orthogonal $F$ -Squares.

From results in Bose (1938) and (1946), we may write the following theorem on the construction of the complete set of orthogonal latin squares from a symmetrical factorial experiment.

**Theorem 2.1.** Let  $n = S^m$  where  $S$  is a prime number and  $m$  is a positive integer. The complete set of orthogonal latin squares of order  $n$ , i.e. the  $OL(n, n-1)$  set, can be constructed from a  $(S^m)^2$  symmetrical factorial experiment, i.e. a symmetrical factorial experiment with 2 factors each at  $S^m$  levels.

A similar theorem for F-squares due to Hedayat, Raghavarao, and Seiden (1975), is:

Theorem 2.2. Let  $S$  be a prime number and  $m$  be a positive integer. For any integer  $p$ , that is a divisor of  $m$ , the complete set of  $(S^m-1)^2/(S^p-1)$  orthogonal  $F(S^m; S^{m-p}, S^{m-p}, \dots, S^{m-p})$ -squares can be constructed from a  $(S^p)^{\frac{2m}{p}}$  symmetrical factorial experiment, i.e. a symmetrical factorial experiment with  $\frac{2m}{p}$  factors each a  $S^p$  levels.

### 3. Decomposing Latin Squares into F-Squares

The following is a theorem on the decomposition of latin squares into orthogonal F-squares.

Theorem 3.1. Each latin square in the set of orthogonal latin squares,  $OL(S^m, S^{m-1})$ , can be decomposed into  $(S^m-1)/(S-1)$  orthogonal  $F(S^m; S^{m-1}, S^{m-1}, \dots, S^{m-1})$ -squares, and the entire  $OL(S^m, S^{m-1})$  set can be decomposed into  $(S^m-1)^2/(S-1)$  orthogonal  $F(S^m; S^{m-1}, S^{m-1}, \dots, S^{m-1})$ -squares.

Proof: Consider a  $(S)^{2m}$  symmetrical factorial experiment, i.e., a symmetrical factorial experiment with  $2m$  factors each at  $S$  levels. Taking  $p = 1$  in theorem 2.2, we can

construct from the unconfounded-with-rows-and-columns pencils  $Q_1, Q_2, \dots, Q_k$  where  $k = (S^m - 1)^2 / (S - 1)$ , the complete set of  $k$  orthogonal  $F(S^m; S^{m-1}, S^{m-1}, \dots, S^{m-1})$ -squares. It is also true from theorem 2.1, since  $S^{2m} = (S^m)^2$ , that we may construct the complete set of  $n - 1 = S^m - 1$  orthogonal latin squares of order  $n = S^m$ , i.e. the  $OL(n, n-1)$  set, from the unconfounded-with-rows-and-columns pencils  $\tilde{Q}_1, \tilde{Q}_2, \dots, \tilde{Q}_{n-1}$  in a symmetrical factorial experiment with 2 factors, each at  $S^m$  levels. Since it is true that each pencil  $\tilde{Q}_i$  is made up of  $(S^m - 1) / (S - 1)$  orthogonal  $Q_j$ 's, we have that each latin square of order  $n = S^m$ , is made up of, or decomposes into,  $(S^m - 1) / (S - 1)$  orthogonal  $F(S^m; S^{m-1}, S^{m-1}, \dots, S^{m-1})$ -squares. Hence we have that the entire  $OL(S^m, S^m - 1)$  set decomposes into  $(S^m - 1)^2 / (S - 1)$  orthogonal  $F(S^m; S^{m-1}, S^{m-1}, \dots, S^{m-1})$ -squares. The second decomposition theorem is a generalization of theorem 3.1:

**Theorem 3.2.** If  $p$  is a divisor of  $m$ , then each latin square in the set of orthogonal latin squares,  $OL(S^m, S^m - 1)$ , can be decomposed into  $(S^m - 1) / (S^p - 1)$  orthogonal  $F(S^m; S^{m-p}, S^{m-p}, \dots, S^{m-p})$ -squares, and the entire  $OL(S^m, S^m - 1)$  set can be decomposed into  $(S^m - 1)^2 / (S^p - 1)$  orthogonal  $F(S^m; S^{m-p}, S^{m-p}, \dots, S^{m-p})$ -squares.

Proof. Consider a  $(S^m)^2$  symmetrical factorial experiment, i.e., a symmetrical factorial experiment with 2 factors each at  $S^m$  levels. By theorem 2.1, we can construct from the unconfounded-with-rows-and-columns pencils  $\tilde{Q}_1, \tilde{Q}_2, \dots, \tilde{Q}_{n-1}$ , where  $n = S^m$ , the complete set of  $n - 1$  orthogonal latin squares of order  $n$ , i.e., the  $OL(n, n-1)$  set. It is also true from theorem 2.2, since  $S^{2m} = (S^P)^{\frac{2m}{P}}$ , that we may construct the complete set of  $k$  orthogonal  $F(S^m; S^{m-p}, S^{m-p}, \dots, S^{m-p})$ -squares, where  $k = (S^m - 1)^2 / (S^P - 1)$ , from the unconfounded-with-rows-and-columns pencils  $Q_1, Q_2, \dots, Q_k$  in a symmetrical factorial experiment with  $\frac{2m}{P}$  factors each at  $S^P$  levels. Since it is true that each pencil  $\tilde{Q}_i$  is made up of  $(S^m - 1) / (S^P - 1)$  orthogonal  $Q_j$ 's, we have that each latin square of order  $n = S^m$ , is made up of, or decomposes into,  $(S^m - 1) / (S^P - 1)$  orthogonal  $F(S^m; S^{m-p}, S^{m-p}, \dots, S^{m-p})$ -squares. Hence we have that the entire  $OL(S^m, S^m - 1)$  set decomposes into  $(S^m - 1)^2 / (S^P - 1)$  orthogonal  $F(S^m; S^{m-p}, S^{m-p}, \dots, S^{m-p})$ -squares.

A third decomposition theorem illustrates how each latin square in an  $OL(S^m, S^m - 1)$  set may be decomposed into  $F$ -squares

with two different numbers of symbols.

**Theorem 3.3.** Each latin square in the set of orthogonal latin squares,  $OL(S^m, S^{m-1})$ , can be decomposed into one  $F(S^m; S, S, \dots, S)$  plus  $S^{m-1} F(S^m; S^{m-1}, S^{m-1}, \dots, S^{m-1})$  orthogonal  $F$ -squares of order  $S^m$ , and the entire  $OL(S^m, S^{m-1})$  set can be decomposed into  $(S^{m-1})F(S^m; S, S, \dots, S)$  plus  $S^{m-1}(S^{m-1})F(S^m; S^{m-1}, S^{m-1}, \dots, S^{m-1})$  orthogonal  $F$ -squares of order  $S^m$ .

**Proof.** Consider a  $(S)^{2m}$  symmetrical factorial experiment, i.e., a symmetrical factorial experiment with  $2m$  factors each at  $S$  levels. Taking  $p = 1$  in theorem 2.2 we can construct from the unconfounded-with-rows-and-columns pencils  $Q_1, Q_2, \dots, Q_k$  where  $k = (S^{m-1})^2 / (S-1)$ , the complete set of  $k$  orthogonal  $F(S^m; S^{m-1}, S^{m-1}, \dots, S^{m-1})$ -squares. There exists  $(S^{m-1})$  sets of  $(S^{m-1}-1)/(S-1)$   $Q_i$ 's that form  $(S^{m-1})$  pencils each with  $S^{m-1} - 1$  degrees of freedom. We can use these to form  $(S^{m-1})$  orthogonal  $F(S^m; S, S, \dots, S)$ -squares. Hence we have formed a set of  $(S^{m-1})F(S^m; S, S, \dots, S)$  plus

$$(S^{m-1})^2 / (S-1) - (S^{m-1}) [(S^{m-1}-1)/(S-1)]$$

$$S^{m-1} (S^{m-1}) F(S^m; S^{m-1}, S^{m-1}, \dots, S^{m-1})$$



orthogonal F-squares of order  $S^m$ .

It is also true from theorem 2.1, since  $S^{2m} = (S^m)^2$ , that we may construct the complete set of  $n - 1 = S^m - 1$  orthogonal latin squares of order  $n = S^m$ , i.e. the  $OL(n, n-1)$  set, from the unconfounded-with-rows-and-columns pencils  $\tilde{Q}_1, \tilde{Q}_2, \dots, \tilde{Q}_{n-1}$  in a symmetrical factorial experiment with 2 factors each at  $S^m$  levels. Since it is true that each pencil  $\tilde{Q}_i$  is made up of  $(S^m-1)/(S-1)$  orthogonal  $Q_j$ 's, we have that each latin square of order  $n = S^m$ , is made up of or decomposes into

$(S^m-1)F(S^m; S, S, \dots, S)$  plus  $S^{m-1}(S^m-1)F(S^m; S^{m-1}, S^{m-1}, \dots, S^{m-1})$  orthogonal F-squares of order  $S^m$ ; we have that the entire  $OL(S^m, S^{m-1})$  set decomposes into  $(S^m-1)F(S^m; S, S, \dots, S)$  plus  $S^{m-1}(S^m-1)F(S^m; S^{m-1}, S^{m-1}, \dots, S^{m-1})$  orthogonal F-squares of order  $S^m$ .

#### 4. Complete Sets of Orthogonal F-Squares with Differing Numbers of Symbols

We show in this section, how to construct complete sets of orthogonal F-squares of order  $n = S^m$ , where the F-squares

in the sets have differing numbers of symbols, instead of a constant number. This is useful to experimenters who have differing numbers of treatments from square to square.

**Theorem 4.1.** There exists and one can construct complete sets of orthogonal F-squares of order  $n = S^m$  where the F-squares are of varying types. In particular, any complete set of orthogonal F-squares of order  $S^m$  can contain  $F(S^m; S^{m-p}, S^{m-p}, \dots, S^{m-p})$ -squares for any integer  $p$ ,  $1 \leq p \leq m$ , that divides  $m$ , and  $F(S^m; S, S, \dots, S)$ -squares.

**Proof.** Take the  $OL(S^m, S^{m-1})$  set. We can decompose as many latin squares as we wish from the OL set into

$F(S^m; S^{m-p}, S^{m-p}, \dots, S^{m-p})$ -squares for integers  $p$  that divide  $m$ , including  $p = 1$  and  $p = m$  by theorem 3.2.

We can also decompose as many latin squares as we wish into  $F(S^m; S, S, \dots, S)$ -squares and  $F(S^m; S^{m-1}, S^{m-1}, \dots, S^{m-1})$ -squares by theorem 3.3.

**Example 4.1.** There exists a complete set of orthogonal F-squares of order  $n = 2^2 = 4$  consisting of

(a)  $9F(4; 2, 2)$ -squares

or

- (b)  $6F(4; 2, 2)$ -squares and  $1F(4; 1, 1, 1, 1)$ -squares  
 or  
 (c)  $3F(4; 2, 2)$ -squares and  $2F(4; 1, 1, 1, 1)$ -squares  
 or  
 (d)  $3F(4; 1, 1, 1, 1)$ -squares.

These four complete sets of orthogonal F-squares of order four are obtained from the  $OL(4,3)$  set. We get set (a) by decomposing all 3 latin squares in the  $OL$  set into  $F(4; 2, 2)$ -squares by Theorem 3.1, set (b) is gotten by decomposing 2 latin squares in the  $OL$  set into  $F(4; 2, 2)$ -squares and leaving the third latin square undecomposed, set (c) is gotten by decomposing 1 latin square in the  $OL$  set into  $F(4; 2, 2)$ -squares and leaving the other two latin squares undecomposed, and set (d) is obtained by leaving all 3 latin squares in the  $OL$  set undecomposed. (One can also use a latin square of order 4 with no orthogonal mate, decompose it into  $3F(4; 2, 2)$ -squares and then can find 6 other  $F(4; 2, 2)$  squares to construct sets (a) or (b).)

Example 4.2. One of the many complete sets of orthogonal F-squares of order  $n = 2^6 = 64$  that exists, consists of  
 $10(63) + 9(32) = 918$   $F(64; 32, 32)$ -squares,  
 $12(21) = 252$   $F(64; 16, 16, 16, 16)$ -squares,  
 $21(9) = 189$   $F(64; 8, 8, \dots, 8)$ -squares,  
 $9(1) = 9$   $F(64; 2, 2, \dots, 2)$ -squares, and  
 $11(1) = 11$   $F(64; 1, 1, \dots, 1)$ -squares.

This set is obtained from the  $OL(64, 63)$  set by decomposing 10 latin squares into  $F(64; 32, 32)$ -squares by theorem 3.2, decomposing 12 latin squares into  $F(64; 16, 16, 16, 16)$ -squares by theorem 3.2, decomposing 21 latin squares into  $F(64; 8, 8, \dots, 8)$ -squares by theorem 3.2, decomposing 9 latin squares into  $F(64; 2, 2, \dots, 2)$ -squares and  $F(64; 32, 32, 32)$ -squares by theorem 3.3, and leaving 11 latin squares undecomposed.

#### 5. Example.

As an example consider the  $OL(2^3, 2^3-1) = OL(8, 7)$  set. We may relate the complete set of 7 orthogonal latin squares of order 8 to a  $2^{2(3)} = 2^6$  factorial treatment design. Let the six main effects be A, B, C, D, E, and F each at two levels 0 and 1. We set up an  $8 \times 8$  square consisting of the  $2^6 = 64$  treatment combinations, confounding three main effects and their interactions with rows and three main effects and their interactions with columns. Without loss of generality let us confound main effects A, B, C and their interactions AB, AC, BC, ABC with rows and let us confound main effects D, E, F and their interactions DE, DF, EF, DEF with columns. Then we have the square in figure 5.1.

		Columns							
Rows		1	2	3	4	5	6	7	8
1		000000	000100	000010	000110	000001	000101	000011	000111
2		100000	100100	100010	100110	100001	100101	100011	100111
3		010000	010100	010010	010110	010001	010101	010011	010111
4		110000	110100	110010	110110	110001	110101	110011	110111
5		001000	001100	001010	001110	001001	001101	001011	001111
6		101000	101100	101010	101110	101001	101101	101011	101111
7		011000	011100	011010	011110	011001	011101	011011	011111
8		111000	111100	111010	111110	111001	111101	111011	111111

Figure 5.1

We obtain the following analysis of variance table relating F-squares and latin squares to the effects in the  $2^6$  factorial treatment design:

		Source of variation	d.f.	
		CFM	1	
		ROWS	7	
		A	1	
		B	1	
		AB	1	
		C	1	
		AC	1	
		BC	1	
		ABC	1	
		COLUMNS	7	
		D	1	
		E	1	
		DE	1	
		F	1	
		DF	1	
		EF	1	
		DEF	1	
		LATIN SQUARE NUMBER ONE TREATMENTS	7	
$F_1(8;2,2,2,2)$ treatments	$\left\{ \begin{array}{l} \text{AD} \\ \text{BE} \\ \text{ABDE} \end{array} \right.$	$= F_1(8;4,4)$ treatments	1	
		$= F_2(8;4,4)$ treatments	1	
		$= F_3(8;4,4)$ treatments	1	
			$= F_4(8;4,4)$ treatments	1
			$= F_5(8;4,4)$ treatments	1
			$= F_6(8;4,4)$ treatments	1
			$= F_7(8;4,4)$ treatments	1

	LATIN SQUARE NUMBER TWO TREATMENTS	7	
$F_2(8;2,2,2,2)$ treatments	{	ADEF = $F_8(8;4,4)$ treatments	1
		BD = $F_9(8;4,4)$ treatments	1
		ABEF = $F_{10}(8;4,4)$ treatments	1
		CDE = $F_{11}(8;4,4)$ treatments	1
		ACF = $F_{12}(8;4,4)$ treatments	1
		BCE = $F_{13}(8;4,4)$ treatments	1
		ABCDF = $F_{14}(8;4,4)$ treatments	1
		}	3
	LATIN SQUARE NUMBER THREE TREATMENTS	7	
$F_3(8;2,2,2,2)$ treatments	{	AEF = $F_{15}(8;4,4)$ treatments	1
		BCF = $F_{16}(8;4,4)$ treatments	1
		ABCE = $F_{17}(8;4,4)$ treatments	1
		ABDF = $F_{18}(8;4,4)$ treatments	1
		BDE = $F_{19}(8;4,4)$ treatments	1
		ACD = $F_{20}(8;4,4)$ treatments	1
		CDEF = $F_{21}(8;4,4)$ treatments	1
		}	3
	LATIN SQUARE NUMBER FOUR TREATMENTS	7	
$F_4(8;2,2,2,2)$ treatments	{	ADF = $F_{22}(8;4,4)$ treatments	1
		ABCF = $F_{23}(8;4,4)$ treatments	1
		BCD = $F_{24}(8;4,4)$ treatments	1
		ABE = $F_{25}(8;4,4)$ treatments	1
		BDEF = $F_{26}(8;4,4)$ treatments	1
		CEF = $F_{27}(8;4,4)$ treatments	1
		ACDE = $F_{28}(8;4,4)$ treatments	1
		}	3

	LATIN SQUARE NUMBER FIVE TREATMENTS	7	
$F_5(8;2,2,2,2)$ treatments	{	AF = $F_{29}(8;4,4)$ treatments	1
		BDF = $F_{30}(8;4,4)$ treatments	1
		ABD = $F_{31}(8;4,4)$ treatments	1
		CE = $F_{32}(8;4,4)$ treatments	1
		ACEF = $F_{33}(8;4,4)$ treatments	1
		BCDEF = $F_{34}(8;4,4)$ treatments	1
		ABCDE = $F_{35}(8;4,4)$ treatments	1
		}	3
	LATIN SQUARE NUMBER SIX TREATMENTS	7	
$F_6(8;2,2,2,2)$ treatments	{	ADE = $F_{36}(8;4,4)$ treatments	1
		BF = $F_{37}(8;4,4)$ treatments	1
		ABDEF = $F_{38}(8;4,4)$ treatments	1
		BCDF = $F_{39}(8;4,4)$ treatments	1
		ABCEF = $F_{40}(8;4,4)$ treatments	1
		CD = $F_{41}(8;4,4)$ treatments	1
		ACE = $F_{42}(8;4,4)$ treatments	1
		}	3
	LATIN SQUARE NUMBER SEVEN TREATMENTS	7	
$F_7(8;2,2,2,2)$ treatments	{	BEF = $F_{43}(8;4,4)$ treatments	1
		BCDE = $F_{44}(8;4,4)$ treatments	1
		CDF = $F_{45}(8;4,4)$ treatments	1
		AE = $F_{46}(8;4,4)$ treatments	1
		ABF = $F_{47}(8;4,4)$ treatments	1
		ABCD = $F_{48}(8;4,4)$ treatments	1
		ACDEF = $F_{49}(8;4,4)$ treatments	1
		}	3
TOTAL		64	

To construct latin square number one from effects AD, BE, ABDE, CF, ACDF, BCEF, and ABCDEF we let the symbols I, II, ..., VIII in the latin square be represented as shown in Figure 5.2 . We now take the  $8 \times 8$  square of the  $2^6$  treatment combinations (Figure 5.1) and put our "treatments" I, II, ..., VIII in the appropriate cells. We then get the following  $8 \times 8$  latin square:

I	V	III	VII	II	VI	IV	VIII
V	I	VII	III	VI	II	VIII	IV
III	VII	I	V	IV	VIII	II	VI
VII	III	V	I	VIII	IV	VI	II
II	VI	IV	VIII	I	V	III	VII
VI	II	VIII	IV	V	I	VII	III
IV	VIII	II	VI	III	VII	I	V
VIII	IV	VI	II	VII	III	V	I

The remaining six latin squares are constructed in the same manner from their corresponding set of seven single degree of freedom effects in the analysis of variance table. The seven latin squares of order 8 constructed in this manner are pairwise orthogonal. Hence we have constructed the  $OL(8,7)$  set from the analysis of variance of the  $2^6$  factorial treatment design.

Each single degree of freedom effect can in turn be used to construct an  $F(8;4,4)$ -square by Theorem 2.2. To construct the F-square  $F_1(8;4,4)$  from the AD effect in the analysis of variance table we let the symbols  $\alpha$  and  $\beta$  in the  $F_1(8;4,4)$ -square



level of Effect

Combinations

AD) <sub>0</sub> , (BE) <sub>0</sub> , (CF) <sub>0</sub> , (ABDE) <sub>0</sub> , (ACDF) <sub>0</sub> , (BCEF) <sub>0</sub> , (ABCDEF) <sub>0</sub>	000000, 100100, 010010, 110110, 001001, 101101, 011011, 111111 =	I
AD) <sub>0</sub> , (BE) <sub>0</sub> , (CF) <sub>1</sub> , (ABDE) <sub>0</sub> , (ACDF) <sub>1</sub> , (BCEF) <sub>1</sub> , (ABCDEF) <sub>1</sub>	001000, 101100, 011010, 111110, 000001, 100101, 010011, 110111 =	II
AD) <sub>0</sub> , (BE) <sub>1</sub> , (CF) <sub>0</sub> , (ABDE) <sub>1</sub> , (ACDF) <sub>0</sub> , (BCEF) <sub>1</sub> , (ABCDEF) <sub>1</sub>	010000, 110100, 000010, 100110, 011001, 111101, 001011, 101111 =	III
AD) <sub>0</sub> , (BE) <sub>1</sub> , (CF) <sub>1</sub> , (ABDE) <sub>1</sub> , (ACDF) <sub>1</sub> , (BCEF) <sub>0</sub> , (ABCDEF) <sub>0</sub>	011000, 111100, 001010, 101110, 010001, 110101, 000011, 100111 =	IV
AD) <sub>1</sub> , (BE) <sub>0</sub> , (CF) <sub>0</sub> , (ABDE) <sub>1</sub> , (ACDF) <sub>1</sub> , (BCEF) <sub>0</sub> , (ABCDEF) <sub>1</sub>	100000, 000100, 110010, 010110, 101001, 001101, 111011, 011111 =	V
AD) <sub>1</sub> , (BE) <sub>0</sub> , (CF) <sub>1</sub> , (ABDE) <sub>1</sub> , (ACDF) <sub>0</sub> , (BCEF) <sub>1</sub> , (ABCDEF) <sub>0</sub>	101000, 001100, 111010, 011110, 100001, 000101, 110011, 010111 =	VI
AD) <sub>1</sub> , (BE) <sub>1</sub> , (CF) <sub>0</sub> , (ABDE) <sub>0</sub> , (ACDF) <sub>1</sub> , (BCEF) <sub>1</sub> , (ABCDEF) <sub>0</sub>	110000, 010100, 100010, 000110, 111001, 011101, 101011, 001111 =	VII
AD) <sub>1</sub> , (BE) <sub>1</sub> , (CF) <sub>1</sub> , (ABDE) <sub>0</sub> , (ACDF) <sub>0</sub> , (BCEF) <sub>0</sub> , (ABCDEF) <sub>1</sub>	111000, 011100, 101010, 001110, 110001, 010101, 100011, 000111 =	VIII

Figure 5.2

be represented as follows:

<u>Level of Effect</u>	<u>Combinations (see Figure 5.2)</u>
(AD) <sub>0</sub>	I, II, III, IV = $\alpha$
(AD) <sub>1</sub>	V, VI, VII, VIII = $\beta$

We now take the  $8 \times 8$  square of the  $2^6$  treatment combinations (Figure 5.1) and put our "treatments"  $\alpha$  and  $\beta$  in the appropriate cells. Or alternatively, we could take the previously constructed latin square number one and replace "treatments" I, II, III, and IV by "treatment"  $\alpha$  and "treatments" V, VI, VII, and VIII by "treatment"  $\beta$ . In either case we get the following  $F_1(8;4,4)$ -square:

$\alpha$	$\beta$	$\alpha$	$\beta$	$\alpha$	$\beta$	$\alpha$	$\beta$
$\beta$	$\alpha$	$\beta$	$\alpha$	$\beta$	$\alpha$	$\beta$	$\alpha$
$\alpha$	$\beta$	$\alpha$	$\beta$	$\alpha$	$\beta$	$\alpha$	$\beta$
$\beta$	$\alpha$	$\beta$	$\alpha$	$\beta$	$\alpha$	$\beta$	$\alpha$
$\alpha$	$\beta$	$\alpha$	$\beta$	$\alpha$	$\beta$	$\alpha$	$\beta$
$\beta$	$\alpha$	$\beta$	$\alpha$	$\beta$	$\alpha$	$\beta$	$\alpha$
$\alpha$	$\beta$	$\alpha$	$\beta$	$\alpha$	$\beta$	$\alpha$	$\beta$
$\beta$	$\alpha$	$\beta$	$\alpha$	$\beta$	$\alpha$	$\beta$	$\alpha$

The remaining forty eight  $F(8;4,4)$ -squares are constructed in the same manner from their corresponding single degree of freedom effect in the analysis of variance table. The forty nine  $F(8;4,4)$ -squares constructed in this manner are pairwise orthogonal. Hence each latin square in the  $OL(8,7)$  set

decomposes into 7 orthogonal  $F(8;4,4)$ -squares and the entire  $OL(8,7)$  set decomposes into 49 orthogonal  $F(8;4,4)$ -squares.

In the preceding analysis of variance table, under Latin Square Number One Treatments, we see that the set of three effects AD, BE, and ABDE is closed under multiplication and hence can be used to construct an  $F_1(8;2,2,2,2)$ -square. To construct this  $F_1(8;2,2,2,2)$ -square we let the symbols W, X, Y, and Z in the  $F_1(8;2,2,2,2)$ -square be represented as follows:

<u>Level of Effect</u>	<u>Combinations (see Figure 5.2)</u>
$(AD)_0, (BE)_0, (ABDE)_0$	I, II = W
$(AD)_0, (BE)_1, (ABDE)_1$	III, IV = X
$(AD)_1, (BE)_0, (ABDE)_1$	V, VI = Y
$(AD)_1, (BE)_1, (ABDE)_0$	VII, VIII = Z

We not take the  $8 \times 8$  square of the  $2^6$  treatment combinations (Figure 5.1) and put our "treatments" W, X, Y, and Z in the appropriate cells. Or alternatively, we could take the previously constructed latin square number one and replace "treatments" I and II by "treatment" W, "treatments" III and IV by "treatment" X, "treatments" V and VI by Y, and "treatments" VII and VIII by Z. In either case we get the following  $F_1(8;2,2,2,2)$ -square:

W	Y	X	Z	W	Y	X	Z
Y	W	Z	X	Y	W	Z	X
X	Z	W	Y	X	Z	W	Y
Z	X	Y	W	Z	X	Y	W
W	Y	X	Z	W	Y	X	Z
Y	W	Z	X	Y	W	Z	X
X	Z	W	Y	X	Z	W	Y
Z	X	Y	W	Z	X	Y	W

Note that the set of seven effects corresponding to each latin square has such a subset of three effects that is closed under multiplication. Hence we see that each latin square in the  $OL(8,7)$  set decomposes into one  $F(8;2,2,2,2)$  and four  $F(8;4,4)$ -squares. And so we can say that the entire  $OL(8,7)$  set decomposes into seven  $F(8;2,2,2,2)$ -squares and twenty-eight  $F(8;4,4)$ -squares. This is a direct application of Theorem 3.3.

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