SOME RESULTS ON ELEMENTARY REFLECTORS

Samuel G. Lindle[1] and David M. Allen[1]

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Abstract

Elementary reflectors are defined and some known fundamental properties are restated for easy reference. A theorem is proved which simplifies to a large extent the process of identifying whether a matrix is an elementary reflector or not. It also provides control over the often needed construction of such matrices. An appendix gives examples of certain sequences of elementary reflectors.

[1] Department of Statistics, University of Kentucky, Lexington, Ky. 40506. This paper was prepared while the authors were on leave at Cornell University.

[2] The Biometrics Unit Mimeo Series, Cornell University, Ithaca, N.Y. 14853.

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1. Introduction. Elementary reflectors (ERs) provide computationally efficient methods of reducing a matrix to triangular or bidiagonal form. Schwarz et al. [1], for example, demonstrates an efficient algorithm for using ERs to tridiagonalize symmetric matrices. Likewise Stewart [2] gives an efficient algorithm for using ERs to triangularize any square matrix. In the statistical solution of least squares, ERs used as premultipliers and/or postmultipliers of the variable matrix can adequately replace the job done by Cholesky factorization and yet provide a more appealing computational approach. The authors have shown in [3] that ERs can be used to provide a new theory of determinants emphasizing geometric interpretation and computation.

2. Definition and Basic Properties. In this paper an $n \times n$ matrix R will be called an ER if R can be written as

$$R = I - 2ww'$$

where w is an $n \times 1$ vector composed of real components and where $w'w = 1$.

It is easy to show that:

a) R is symmetric.

b) R is an orthogonal matrix and consequently preserves both norms and angles when used to premultiply (postmultiply) a matrix of column (row) vectors.

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c) R has on eigenvalue equal to -1 and the rest are all ones. (One can alternately define an ER as any symmetric matrix with this property.)

d) The eigenvectors of R are the columns of $Q = [w|P]$ where P is an n x n-1 matrix with orthonormal columns and with all the columns orthogonal to w . Here w is the eigenvector associated with the -1 eigenvalue and the n-1 columns of P are the eigenvectors associated with the n-1 eigenvalues that are equal to 1 .

e) R has determinant equal to -1 .

If we let $w = x/\|x\|$ in our definition of ERs we have

$$R = I - 2x(x'x)^{-1} x' \quad .$$

The projection matrix

$$P = x(x'x)^{-1} x$$

can be used to see the motivation in the term "elementary reflector", for

$$R = I - 2P = (I-P) - P \quad .$$

Thus, referring to figure 1, if y is any real valued vector then it is well known that the projection of y onto x is Py and the projection of y onto the space orthogonal to x is (I-P)y . Since R is orthogonal and equals the difference (I-P) - P, it acts as an angle preserving, length preserving <u>reflection</u> of y about (I-P)y as is indicated in figure 1. The reflection is <u>elementary</u> in the sense that it occurs only in a single plane. In this case the plane is the one defined by the vectors x and y .

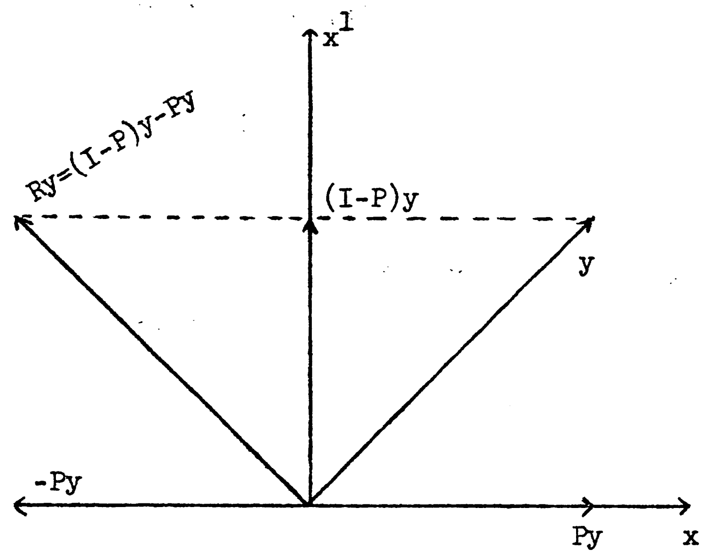


Figure 1.

Pictorial Illustration of an Elementary Reflection

Some ERs are of particular interest. First, suppose

$$x_i = 1$$
$$x_j = -1 \qquad j \neq i$$
$$x_k = 0 \qquad k \neq i,j \; ; \qquad k=1,\cdots,n$$

then Ry permutes the i'th and j'th elements of y . We denote an ER of this type by P_{ij} . (P_{ij} is often called an elementary permutation matrix.)

If $x_i = \pm 1$ and $x_k = 0$ for all $k \neq i$ then Ry changes the sign of the i'th element.

If $x_i = 0 \qquad i=1,\cdots,k-1$

$$x_k = y_k + \text{sign}\,(y_k)\sqrt{\sum_{i=k}^{n} y_i^2}$$

$$x_i = y_i \qquad i=k+1,\cdots,n$$

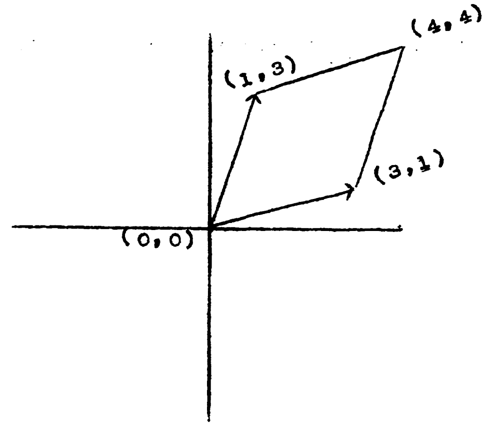
then Ry retains the first k-1 elements, the k'th element is

$$- \text{sign}\ (y_k) \sqrt{\sum_{i=k}^{n} y_i^2}$$

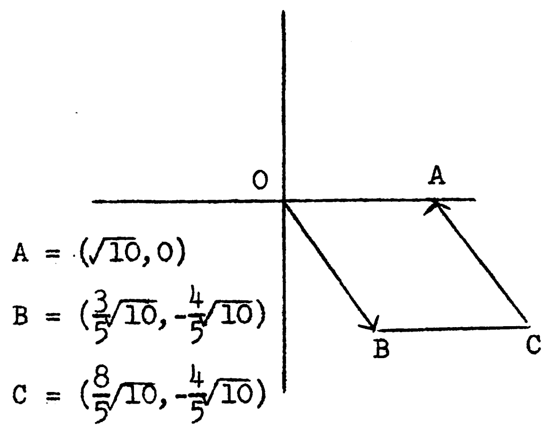
and all other elements are zero. A sequence of ERs of this type can be used to triangularize any square matrix. Figure 2 illustrates this pictorially on the 2 x 2 symmetric matrix

$$\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} .$$

The first premultiplication by an appropriate ER triangularizes the matrix and the second puts the figure in the first quadrant. Two sequences, one for premultiplication and the other for postmultiplication, of ERs of this type can be used to tridiagonalize or bidiagonalize any square matrix. ERs are numerically efficient operators. Details with regard to ERs and their numerical properties are given in Schwarz [1] on pages 139-144.



Original Parallelogram

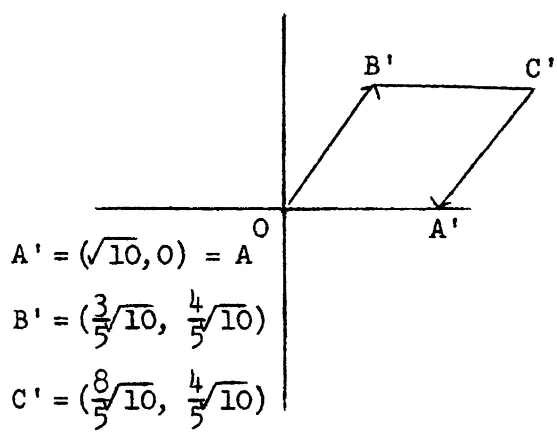


$A = (\sqrt{10}, 0)$

$B = (\frac{3}{5}\sqrt{10}, -\frac{4}{5}\sqrt{10})$

$C = (\frac{8}{5}\sqrt{10}, -\frac{4}{5}\sqrt{10})$

After One Premultiplication by Appropriate Elementary Reflector



$A' = (\sqrt{10}, 0) = A$

$B' = (\frac{3}{5}\sqrt{10}, \frac{4}{5}\sqrt{10})$

$C' = (\frac{8}{5}\sqrt{10}, \frac{4}{5}\sqrt{10})$

Final Reflection About X Axis to Put the Figure in the First Quadrant

Figure 2.

Two-Dimensional Representation of the Effect of Premultiplication of a Matrix Successively by Elementary Reflectors so as to Reduce it to an Upper Triangular Matrix.

3. An Identification Theorem. The following theorem is helpful in identifying whether a symmetric matrix is an ER or not and in constructing ERs when they are needed.

Identification Theorem. *Any symmetric $n \times n$ matrix A is an ER if and only if the rows of $A-I$ are proportional and $tr(A-I) = -2$.*

Pf. For the "if" part we have that since A is symmetric, so is $A-I$. Also, since the rows of $A-I$ are proportional, rank $(A-I) = 1$. Thus, using spectral decomposition, there exists a unit vector w all of whose components are real such that

$$A - I = kww'$$

for some scalar k . But $tr(A-I) = -2$ implies that $k = -2$. Consequently, $A = I-2ww'$ is an ER.

For the "only if" part since A is an ER it can be written as $A = I-2ww'$ where $w'w = 1$ and all the elements of w are real. Thus $A-I= -2ww'$. But the i'th row of $A-I$ is $r_i = -2w_i w'$ for all $i=1,\cdots,n$. Consequently, the rows of $A-I$ are proportional. Also $tr(A-I) = tr(-2ww') = -2tr(w'w) = -2$.

Note that this theorem requires $n(n-1)$ multiplications and divisions in order to verify that a matrix is an ER. Fewer multiplications and divisions will usually be required to verify that a matrix is not an ER. If one wishes to construct an $n \times n$ ER once can begin by constructing any $n \times n$ symmetric matrix B whose rows are proportional, find its trace and multiply the matrix by $-2/tr(B)$ and add the identity matrix to the result.

BIBLIOGRAPHY

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[2] G. W. Stewart. Introduction to Matrix Computations, Academic Press, New York and London, 1973.

[3] S. G. Lindle and D. M. Allen. "A New Theory of Determinants Emphasizing Geometric Interpretation and Computation". BU-534-M in the Biometrics Unit Mimeo Series, Cornell University, Ithaca, N.Y. 14853, October 1974.

Appendix: Some Special ERs.

Standard Sequence

$$-1,\ \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix},\ \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ \ldots,\ \begin{bmatrix} -1 & & & & & 0 \\ & 1 & & & & \\ & & 1 & & & \\ & & & \ddots & & \\ & * & & & \ddots & \\ & & & & & 1 \end{bmatrix},\ \ldots$$

Permuted Sequence

$$-1,\ \begin{bmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{bmatrix},\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & \pm 1 \\ 0 & \pm 1 & 0 \end{bmatrix},\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \pm 1 \\ 0 & 0 & \pm 1 & 0 \end{bmatrix},\ \ldots,\ \begin{bmatrix} 1 & & & & & & 0 \\ & 1 & & & & & \\ & & \ddots & & & & \\ & & & \ddots & & & \\ & * & & & 1 & & \\ & & & & & 0 & \pm 1 \\ & & & & & & 0 \end{bmatrix},\ \ldots$$

Equal Diagonal Sequence

$$-1,\ \begin{bmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{bmatrix},\ \frac{1}{3}\begin{bmatrix} 1 & \pm 2 & \pm 2 \\ * & 1 & \pm 2 \\ & & 1 \end{bmatrix},\ \frac{1}{2}\begin{bmatrix} 1 & \pm 1 & \pm 1 & \pm 1 \\ & 1 & \pm 1 & \pm 1 \\ & & 1 & \pm 1 \\ & & & 1 \end{bmatrix},\ \frac{1}{5}\begin{bmatrix} 3 & \pm 2 & \pm 2 & \pm 2 & \pm 2 \\ & 3 & \pm 2 & \pm 2 & \pm 2 \\ & & 3 & \pm 2 & \pm 2 \\ * & & & 3 & \pm 2 \\ & & & & 3 \end{bmatrix},$$

$$\ldots\ \frac{1}{k}\begin{bmatrix} k-2 & \pm 2 & \cdots & \pm 2 \\ \vdots & & & \vdots \\ * & \ddots & & \vdots \\ & & \ddots & k-2 \end{bmatrix},\ \ldots\ .$$

$*$ = symmetric