A DECOMPOSITION OF A LATIN SQUARE OF ORDER 6 INTO F-SQUARES AND SOME OBSERVATIONS ON ORTHOGONALITY IN F-SQUARES

BU-549-M | by | February, 1975

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Abstract

A given latin square of order 6 is decomposed into two sets of single degree of freedom contrasts and resulting F-squares are obtained from the single degree of freedom contrasts. The resulting squares are considered in terms of the published definition (by A. Hedayat) of F-squares and found lacking. A definition of orthogonal F-squares for unordered pairs is given and several sets of four $F(1^3,1^3;6)$-squares can be found to satisfy this definition. It is considered impossible to obtain a pair of orthogonal $F(1^3,1^3;6)$-squares using Hedayat's definition.

Paper No. BU-549-M in the Biometrics Unit Mimeo Series, Department of Plant Breeding and Biometry, Cornell University, Ithaca, New York 14853.

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1. A 6 X 6 Latin Square

Consider the following latin square of order 6, which may be constructed as the Kronecker product of a latin square of order 2 and one of order 3:

$$\begin{array}{|cc|}\hline 0 & 1 \\ 1 & 0 \\ \hline \end{array} \otimes \begin{array}{|ccc|}\hline 0 & 1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \\ \hline \end{array} = \begin{array}{|ccc:ccc|}\hline A & B & C & D & E & F \\ B & C & A & E & F & D \\ C & A & B & F & D & E \\ \hdashline D & E & F & A & B & C \\ E & F & D & B & C & A \\ F & D & E & C & A & B \\ \hline \end{array}$$

Construct a set of orthogonal contrasts among the letters as follows:

	Letters					
Contrast	A	B	C	D	E	F
C1	1	1	1	1	1	1
C2	1	1	1	-1	-1	-1
C3	1	1	-2	2	-1	-1
C4	-1	1	0	0	-1	1
C5	-2	1	1	1	1	-2
C6	0	1	-1	-1	1	0

Note that the above set of contrasts was selected starting with C2 since this contrast immediately forms an $F(1^3,1^3;6)$-square where 1^3 means that a symbol occurs 3 times in rows and 3 times in columns, the number of one's gives the number of symbols, and 6 is the order of the square. Thus, if all letters receiving a +1 in

C2 are denoted as one, and if all letters receiving a -1 in C2 are designated as zero, then the following results:

$$F_{C2}(1^3,1^3;6) = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right]$$

Likewise, if we place a one where the plus signs occur and a zero where the minus signs occur in C3, we obtain the following F-square:

$$F_{C3}(1^3,1^3;6) = \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ \hline 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \end{array}\right]$$

Similarly, from C5 we obtain:

$$F_{C5}(1^4,1^2;6) = \left[\begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ \hline 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \end{array}\right]$$

where the symbol 1 occurs 4 times in each row and column, and the symbol zero occurs twice in each row and column.

The remaining two contrasts result in rather peculiar F-squares in that $\frac{1}{3}$ of the cells are blank. The "F-squares" are:

$$F_{C4}(1^2,1^2,-;6) = \left[\begin{array}{ccc|ccc} 0 & 1 & - & - & 0 & 1 \\ 1 & - & 0 & 0 & 1 & - \\ - & 0 & 1 & 1 & - & 0 \\ \hline - & 0 & 1 & 0 & 1 & - \\ 0 & 1 & - & 1 & - & 0 \\ 1 & - & 0 & - & 0 & 1 \end{array}\right] \text{ and } F_{C6}(1^2,1^2,-;6) = \left[\begin{array}{ccc|ccc} - & 1 & 0 & 0 & 1 & - \\ 1 & 0 & - & 1 & - & 0 \\ 0 & - & 1 & - & 0 & 1 \\ \hline 0 & 1 & - & - & 1 & 0 \\ 1 & - & 0 & 1 & 0 & - \\ - & 0 & 1 & 0 & - & 1 \end{array}\right]$$

At this point let us count the number of occurrences of symbols with each other, where the pair (i,j) denotes the symbol i in the first square and the symbol j in the second square:

Square (i) vs. Square (j)	(i,j) (0,0)	(0,1)	(1,0)	(1,1)
C2 vs. C3	12	6	6	12
C4	6	6	6	6
C5	6	12	6	12
C6	6	6	6	6
C3 vs. C4	6	6	6	6
C5	6	12	6	12
C6	6	6	6	6
C4 vs. C5	6	6	6	6
C6	0	6	0	6
C5 vs. C6	0	0	12	12

The question now arises about the orthogonality of the above five F-squares. In other words, what is meant by orthogonality of F-squares and is the above method appropriate for constructing F-squares? The treatment contrasts do form an orthogonal set.

2. Treatment Contrasts Using Orthogonal Polynomials

Using orthogonal polynomial coefficients, the contrasts among the six letters are:

Contrast	Letters A	B	C	D	E	F
mean	1	1	1	1	1	1
1 = linear (-)	5	3	1	-1	-3	-5
2 = quadratic	5	-1	-4	-4	-1	5
3 = cubic	-5	7	4	-4	-7	5
4 = quartic	1	-3	2	2	-3	1
5 = quintic	-1	5	-10	10	-5	1

If we now insert a one wherever a positive coefficient occurs and a zero wherever a negative coefficient occurs for the letter in the above latin square, we obtain the following five F-squares:

$$F_1(1^3,1^3;6) = \left[\begin{array}{ccc|ccc} 1&1&1&0&0&0\\ 1&1&1&0&0&0\\ 1&1&1&0&0&0\\ \hline 0&0&0&1&1&1\\ 0&0&0&1&1&1\\ 0&0&0&1&1&1 \end{array}\right], \quad F_2(1^2,1^4;6) = \left[\begin{array}{ccc|ccc} 1&0&0&0&0&1\\ 0&0&1&0&1&0\\ 0&1&0&1&0&0\\ \hline 0&0&1&1&0&0\\ 0&1&0&0&0&1\\ 1&0&0&0&1&0 \end{array}\right],$$

$$F_3(1^3,1^3;6) = \left[\begin{array}{ccc|ccc} 0&1&1&0&0&1\\ 1&1&0&0&1&0\\ 1&0&1&1&0&0\\ \hline 0&0&1&0&1&1\\ 0&1&0&1&1&0\\ 1&0&0&1&0&1 \end{array}\right], \quad F_4(1^4,1^2;6) = \left[\begin{array}{ccc|ccc} 1&0&1&1&0&1\\ 0&1&1&0&1&1\\ 1&1&0&1&1&0\\ \hline 1&0&1&1&0&1\\ 0&1&1&0&1&1\\ 1&1&0&1&1&0 \end{array}\right],$$

$$\text{and } F_5(1^3,1^3;6) = \left[\begin{array}{ccc|ccc} 0&1&0&1&0&1\\ 1&0&0&0&1&1\\ 0&0&1&1&1&0\\ \hline 1&0&1&0&1&0\\ 0&1&1&1&0&0\\ 1&1&0&0&0&1 \end{array}\right].$$

As before, let us count the frequency of occurrences (i,j) for all pairs of the five squares above.

Square (i) vs. Square (j)	(i,j) (0,0)	(0,1)	(1,0)	(1,1)
F_1 vs. F_2	12	6	12	6
F_3	12	6	6	12
F_4	6	12	6	12
F_5	6	12	12	6
F_2 vs. F_3	12	12	6	6
F_4	12	12	0	12
F_5	12	12	6	6
F_3 vs. F_4	6	12	6	12
F_5	12	6	6	12
F_4 vs. F_5	6	6	12	12

3. Orthogonality in F-Squares

Hedayat [1969] gives the following definition for mutually orthogonal F-squares:

Definition 3.1. Given an F-square $F_1(\lambda_1, \lambda_2, \cdots, \lambda_k; n)$ on a k-set $\Sigma = \{a_1, a_2, \cdots, a_k\}$ and an F-square $F_2(\mu_1, \mu_2, \cdots, \mu_t; n)$ on a t-set $\Omega = \{b_1, b_2, \cdots, b_t\}$, we say that F_2 is an orthogonal mate for F_1 (and write $F_1 \perp F_2$) if upon superposition of F_2 on F_1, a_i with frequency in rows and columns of λ_i in F_1 appears $\lambda_i\mu_j$ times with b_j which occurs with frequency μ_j in the rows and in the columns of F_2.

To illustrate the above definition, consider the following set of three mutually orthogonal F-squares of order n = 5 (the superscripts of the symbols are the λ_i and μ_j):

$F_1(1^2, 1^2, 1; 5)$ ⊥ $F_2(1, 1, 1^3; 5)$ ⊥ $F_3(1, 1, 1, 1^2; 5)$

F_1				
1	2	3	1	2
2	1	2	3	1
1	2	1	2	3
3	1	2	1	2
2	3	1	2	1

F_2				
1	2	3	3	3
3	3	1	2	3
2	3	3	3	1
3	1	2	3	3
3	3	3	1	2

F_3				
1	2	3	4	4
3	4	4	1	2
4	1	2	3	4
2	3	4	4	1
4	4	1	2	3

Counting occurrences of symbols in one square with those in another, we obtain:

	F_2		
F_1	1	2	3
1	2	2	6
2	2	2	6
3	1	1	3

	F_3			
F_1	1	2	3	4
1	2	2	2	4
2	2	2	2	4
3	1	1	1	2

	F_3			
F_2	1	2	3	4
1	1	1	1	2
2	1	1	1	2
3	3	3	3	6

Note that the ratio of $\mu_1 : \mu_2 : \mu_3 : \cdots : \mu_t$ remains constant in each row and that the ratio of $\lambda_1 : \lambda_2 : \lambda_3 : \cdots : \lambda_k$ remains constant in each column. Note also that these are the superscripts in the F-squares involved and that the entries in the table are $\lambda_i \mu_j$.

As a second example, consider the following two F-squares of order 6:

$F_1(1^2, 1^2, 1^2; 6)$ ⊥ $F_2(1^2, 1^2, 1^2; 6)$

F_1					
A	B	C	A	B	C
B	C	A	B	C	A
C	A	B	C	A	B
A	B	C	A	B	C
B	C	A	B	C	A
C	A	B	C	A	B

F_2					
A	B	C	A	B	C
C	A	B	C	A	B
B	C	A	B	C	A
A	B	C	A	B	C
C	A	B	C	A	B
B	C	A	B	C	A

The frequencies of occurrence of symbols in the two squares are:

F_1	F_2 A	B	C
A	4	4	4
B	4	4	4
C	4	4	4

It should be noted that Anderson, Federer, and Seiden [1974] were able to obtain a set of 8 mutually orthogonal $F(1^2,1^2,1^2;6)$-squares and that no additional $F(1^2,1^2,1^2;6)$ could be added to this set, which means that the set is <u>locked</u> in the jargon of this area.

Most of the pairs of F-squares presented thus far do not fit this definition. In fact, the following is conjectured:

It is impossible to obtain two orthogonal (in the sense of definition 3.1) $F(1^3,1^3;6)$-squares. For example, consider the $F_{C2}(1^3,1^3;6)$-square. Note that F_{C2} is orthogonal to a peculiar sort of "F-square":

$F_{C2}(1^3,1^3;6)$

1	1	1	0	0	0
1	1	1	0	0	0
1	1	1	0	0	0
0	0	0	1	1	1
0	0	0	1	1	1
0	0	0	1	1	1

"F( )"

1	1	0	1	1	0
1	0	1	1	0	1
0	1	1	0	1	1
0	0	1	0	0	1
0	1	0	0	1	0
1	0	0	1	0	0

The occurrences of symbols in the two squares are:

F_{C2}	"F" 0	1
0	9	9
1	9	9

Note that in "F" the symbol one occurs 4 times in rows 1, 2, and 3 and twice in the remaining rows. In columns each symbol occurs 3 times. Hence symbols and columns are orthogonal, rows and columns are orthogonal, but rows and symbols are not orthogonal. It is considered impossible to find two orthogonal $F(1^3,1^3;6)$ squares of the nature of F_{C2}, and to obtain frequencies of 9 as in the above table.

If we were to alter the definition of orthogonality as follows, we can obtain many sets of mutually orthogonal $F(1^3,1^3;6)$-squares.

<u>Definition 3.2</u>. Two F-squares are considered to be <u>orthogonal for unordered pairs</u> if $\mu_i\lambda_j + \mu_j\lambda_j = \mu_i\mu_i = \lambda_j\lambda_j$ for all i and j as in definition 3.1.

As an example, consider the following four $F(1^3,1^3;6)$-squares:

$F_1(1^3,1^3;6) = F_{C2}$

1	1	1	0	0	0
1	1	1	0	0	0
1	1	1	0	0	0
0	0	0	1	1	1
0	0	0	1	1	1
0	0	0	1	1	1

$F_2(1^3,1^3;6)$

1	1	0	0	0	1
1	0	1	0	1	0
0	1	1	1	0	0
0	0	1	1	1	0
0	1	0	1	0	1
1	0	0	0	1	1

$F_3(1^3,1^3;6)$

0	1	1	1	0	0
1	1	0	0	0	1
1	0	1	0	1	0
0	0	1	1	1	0
0	1	0	1	0	1
1	0	0	0	1	1

$F_4(1^3,1^3;6)$

1	0	1	0	1	0
0	1	1	1	0	0
1	1	0	0	0	1
0	0	1	1	1	0
0	1	0	1	0	1
1	0	0	0	1	1

The occurrences of symbols in one square with those in another are given below:

Pair of squares	Occurrences (1,1)	(0,0)	(0,1) and (1,0)
F_1 vs. F_2	12	12	6 + 6 = 12
F_3	12	12	6 + 6 = 12
F_4	12	12	6 + 6 = 12
F_2 vs. F_3	12	12	6 + 6 = 12
F_4	12	12	6 + 6 = 12
F_3 vs. F_4	12	12	6 + 6 = 12

Thus here is a set of 4 mutually orthogonal $F(1^3,1^3;6)$-squares for unordered pairs. Several such sets are available upon noting the nature of the construction of the above F-squares:

$$F_2 = \begin{array}{|cc|} A_1 & B_1 \\ B_1 & A_1 \end{array} \qquad F_3 = \begin{array}{|cc|} A_2 & B_2 \\ B_1 & A_1 \end{array} \qquad F_4 = \begin{array}{|cc|} A_3 & B_3 \\ B_1 & A_1 \end{array}$$

There are many sets of A_i and B_j and these can be combined in several ways. It was not possible to obtain more than four such sets.

It is possible to obtain at least 8 mutually orthogonal $F(1^2,1^4;6)$-squares in the sense of definition 3.1. These may be obtained from the 8 $F(1^2,1^2,1^2;6)$-squares of Anderson, Federer, and Seiden [1974]. The procedure is to set all 2 elements equal to one and to leave the zero and one elements in their original form.

It is known that F_{C2} in the form $\begin{pmatrix} 1&1&1 \\ 1&1&1 \\ 1&1&1 \end{pmatrix} \times \begin{pmatrix} 1&0 \\ 0&1 \end{pmatrix}$ is orthogonal to the 8 $F(1^2,1^2,1^2;6)$-squares. It is not known if there are additional $F(1^2,1^4;6)$-squares and $F(1,1^5;6)$-squares which are mutually orthogonal to the 8 and to F_{C2}. An attempt is being made to find such squares. Since $F_{C5}(1^4,1^2;6)$ is orthogonal to $F_{C2}(1^3,1^3;6)$ in the sense of definition 3.1, it is a candidate for the above search.

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