A NOTE ON A COVARIANCE MODEL FOR THE 2-WAY CROSSED CLASSIFICATION

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Abstract

A model is considered which allows the coefficient of the covariable to be a sum of two coefficients, one due to row and one due to column.

The familiar model for an observation y_{ijk} in the i^{th} row and j^{th} column of a 2-way cross-classification with interaction is

$$y_{ijk} = \mu + \alpha_i + \beta_j + \gamma_{ij} + e_{ijk}, \tag{0}$$

where μ is a general mean, α_i is the effect due to the i^{th} row, β_j is the effect due to the j^{th} column, γ_{ij} is the corresponding interaction effect, and e_{ijk} is the random error term. The number of rows and columns is denoted by a and b, respectively, so that $i = 1, 2, \cdots, a$ and $j = 1, 2, \cdots, b$, with n_{ij} observations in the (i,j) cell, so that $k = 1, 2, \cdots, n_{ij}$. The possibility that some n_{ij}'s may be zero is not excluded.

The covariance model considered here is

$$y_{ijk} = \mu + \alpha_i + \beta_j + \gamma_{ij} + (b_i^* + b_j)z_{ijk} + e_{ijk} \tag{1}$$

where z_{ijk} is the observed value of the covariate corresponding to y_{ijk}.

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Notation

The model (1) is suggested by Searle [1971, p. 360], hereafter referred to simply as LM. There, however, b_i is used instead of b_i^* of (1), a notation that fails to distinguish for example between b_i for $i = 1$ and b_j for $j = 1$, a distinction that is imperative and that is achieved through using b_i^*. Furthermore, the model (1) also requires having the z-term, the covariable corresponding to y_{ijk}, to be z_{ijk} and not z_{ij} as erroneously shown in LM.

The model

Model (1) is a special case of the general covariance model considered in LM, namely

$$\underline{y} = \underline{X}\underline{a} + \underline{Z}\underline{b} + \underline{e}, \tag{2}$$

where $\underline{y}$ is the vector of observations, $\underline{a}$ is the vector of effects for the design part of the model (in this case μ, the α's, β's and γ's) and $\underline{X}$ is the corresponding design matrix. $\underline{Z}$ is the matrix of observed covariables and $\underline{b}$ is the corresponding vector of coefficients of those covariables. For (1), $\underline{X}$ of (2) is the familiar design matrix for a 2-way classification with interaction; $\underline{Z}$, however, turns out to be singular, as has been pointed out by Zinger [1974]. For example, consider the case of $a = 3$, $b = 2$ and $n_{ij} = 2$ for all i and j. Then $\underline{Z}\underline{b}$ of (2) is, for (1),

$$\underline{Z}\underline{b} = \begin{bmatrix} z_{111} & \cdot & \cdot & z_{111} & \cdot \\ z_{112} & \cdot & \cdot & z_{112} & \cdot \\ z_{121} & \cdot & \cdot & \cdot & z_{121} \\ z_{122} & \cdot & \cdot & \cdot & z_{122} \\ \cdot & z_{211} & \cdot & z_{211} & \cdot \\ \cdot & z_{212} & \cdot & z_{212} & \cdot \\ \cdot & z_{221} & \cdot & \cdot & z_{221} \\ \cdot & z_{222} & \cdot & \cdot & z_{222} \\ \cdot & \cdot & z_{311} & z_{311} & \cdot \\ \cdot & \cdot & z_{312} & z_{312} & \cdot \\ \cdot & \cdot & z_{321} & \cdot & z_{321} \\ \cdot & \cdot & z_{322} & \cdot & z_{322} \end{bmatrix} \begin{bmatrix} b_1^* \\ b_2^* \\ b_3^* \\ b_1 \\ b_2 \end{bmatrix}, \tag{3}$$

where the dots represent zeros. The matrix $\underline{Z}$ is clearly of rank one less than its number of columns, since columns corresponding to all the b_j's sum to the same column vector as do the columns corresponding to all the b_i^*'s. This is also true in the general case of $k = 1, 2, \cdots, n_{ij}$, for unequal n_{ij} including maybe empty cells. The model as it stands therefore violates the assumption made 3 lines below (6) on page 341 of LM, that $\underline{Z}$ should have full column rank. Non-singularity of $\underline{Z}'\underline{Z}$ can, however, be assured, by deleting a column of $\underline{Z}$ as implied in Zinger (op. cit). This is tantamount to putting the corresponding b_i^* or b_j equal to zero. Suppose we put $b_1^* = 0$. Then in the example the coefficients of the covariable become as follows:

	Column 1	Column 2
Row 1	b_1	b_2
Row 2	$b_2^* + b_1$	$b_2^* + b_2$
Row 3	$b_3^* + b_1$	$b_3^* + b_2$

This is certainly reasonable; and generalization is clear.

Estimation

Estimation of $\underline{b}$ in (2) depends (LM p. 343) on calculating $\underline{R}_z$, the matrix of deviations of the z's obtained by fitting $E(\underline{z}) = \underline{X}\underline{a} + \underline{\epsilon}$ for each column $\underline{z}$ of $\underline{Z}$. Recall that for (0) a solution of the normal equations is (LM p. 291)

$$[\mu^o = 0 \quad \underline{\alpha}^{o'} = 0 \quad \underline{\beta}^{o'} = 0 \quad \{\gamma_{ij} = \bar{y}_{ij.}\}]. \tag{4}$$

Applying this in turn to each column of $\underline{Z}$, after deleting its first column, we get, for $\underline{Z}$ of (3),

$$\underset{\sim}{R}_z = \begin{bmatrix} \cdot & \cdot & z_{111} - \bar{z}_{11\cdot} & \cdot \\ \cdot & \cdot & z_{112} - \bar{z}_{11\cdot} & \cdot \\ \cdot & \cdot & \cdot & z_{121} - \bar{z}_{12\cdot} \\ \cdot & \cdot & \cdot & z_{122} - \bar{z}_{12\cdot} \\ z_{211} - \bar{z}_{21\cdot} & \cdot & z_{211} - \bar{z}_{21\cdot} & \cdot \\ z_{212} - \bar{z}_{21\cdot} & \cdot & z_{212} - \bar{z}_{21\cdot} & \cdot \\ z_{221} - \bar{z}_{22\cdot} & \cdot & \cdot & z_{221} - \bar{z}_{22\cdot} \\ z_{222} - \bar{z}_{22\cdot} & \cdot & \cdot & z_{222} - \bar{z}_{22\cdot} \\ \cdot & z_{311} - \bar{z}_{31\cdot} & z_{311} - \bar{z}_{31\cdot} & \cdot \\ \cdot & z_{312} - \bar{z}_{31\cdot} & z_{312} - \bar{z}_{31\cdot} & \cdot \\ \cdot & z_{321} - \bar{z}_{32\cdot} & \cdot & z_{321} - \bar{z}_{32\cdot} \\ \cdot & z_{322} - \bar{z}_{32\cdot} & \cdot & z_{322} - \bar{z}_{32\cdot} \end{bmatrix} \qquad (5)$$

Then

$$\underset{\sim}{R}'_z \underset{\sim}{R}_z = \begin{bmatrix} s_{21} + s_{22} & 0 & s_{21} & s_{22} \\ 0 & s_{31} + s_{32} & s_{31} & s_{32} \\ s_{12} & s_{13} & s_{11} + s_{21} + s_{31} & 0 \\ s_{22} & s_{23} & 0 & s_{12} + s_{22} + s_{32} \end{bmatrix} \qquad (6)$$

where

$$s_{ij} = \sum_{k=1}^{n_{ij}} (z_{ijk} - \bar{z}_{ij\cdot})^2 \; . \qquad (7)$$

And in general

$$\underset{\sim}{R}'_z \underset{\sim}{R}_z = \begin{bmatrix} \underset{\sim}{D}\{s_{i\cdot}\} & \{s_{ij}\} \\ \{s_{ij}\}' & \underset{\sim}{D}\{s_{\cdot j}\} \end{bmatrix} \quad \begin{array}{l} \text{for } i = 2, \cdots, a \\ \quad\; j = 1, 2, \cdots, b \end{array} \qquad (8)$$

$$\overset{\text{def}}{\equiv} \begin{bmatrix} \underline{D}_1 & \underline{C} \\ \underline{C}' & \underline{D}_2 \end{bmatrix} \tag{9}$$

where $\underline{D}\{a_i\}$ for i = 1, 2, $\cdots$, t is a diagonal matrix of order t, its diagonal elements being $a_1, a_2, \cdots, a_t$; and, in the usual manner of summation notation,

$$s_{i.} = \sum_{\substack{j=1 \\ \text{for } n_{ij}\neq 0}}^{b} s_{ij} \quad \text{and} \quad s_{.j} = \sum_{\substack{i=1 \\ \text{for } n_{ij}\neq 0}}^{a} s_{ij} , \tag{10}$$

where these sums are over only those cells containing data, i.e., for which $n_{ij} \neq 0$. Furthermore

$$\underline{R}_z'\underline{y} = \begin{bmatrix} \{p_{i.}\} \\ \{p_{.j}\} \end{bmatrix} \quad \text{for } i = 2, \cdots, a \quad \text{and} \quad j = 1, 2, \cdots, b \tag{11}$$

where

$$p_{ij} = \sum_{k=1}^{n_{ij}} (z_{ijk} - \bar{z}_{ij.})(y_{ijk} - \bar{y}_{ij.}) \tag{12}$$

and $p_{i.}$ and $p_{.j}$ of (11) are sums of sums of products of y's and z's analogous to the diagonal elements of $\underline{R}_z'\underline{R}_z$ in (8). From (9) the two alternative forms of the inverse of $\underline{R}_z'\underline{R}_z$ are

$$(\underline{R}_z'\underline{R}_z)^{-1} = \begin{bmatrix} \underline{D}_1^{-1} & \underline{0} \\ \underline{0} & \underline{0} \end{bmatrix} + \begin{bmatrix} -\underline{D}_1^{-1}\underline{C} \\ \underline{I} \end{bmatrix} (\underline{D}_2 - \underline{C}'\underline{D}_1^{-1}\underline{C})^{-1} [-\underline{C}'\underline{D}_1^{-1} \quad \underline{I}]$$

and

$$= \begin{bmatrix} \underline{0} & \underline{0} \\ \underline{0} & \underline{D}_2^{-1} \end{bmatrix} + \begin{bmatrix} \underline{I} \\ -\underline{D}_2^{-1}\underline{C}' \end{bmatrix} (\underline{D}_1 - \underline{C}\underline{D}_2^{-1}\underline{C}')^{-1} [\underline{I} \quad -\underline{C}\underline{D}_2^{-1}],$$

but neither of them appear to lead to any further simplification of the estimation procedure, which then proceeds in the usual manner (LM pp. 340-361).

Allied models

The model (0) contains interaction terms γ_{ij}. When these are not included, the general solution vector of the normal equations is not as simple as it is for (0). But, when fitting the $\mathbf{z}$-vectors that are columns of $\mathbf{Z}$, e.g., of (3), the solutions will be simple because each $\mathbf{z}$-vector has non-zero elements corresponding only to a single row or a single column of the design. As a result, the solution vector is null except for $\bar{z}_{ij.}$ corresponding to the non-zero elements in $\mathbf{z}$. Hence $\mathbf{R}_z$ is the same as previously, e.g., as in (5). Thus for the no-interaction form of (1), the $\mathbf{R}_z$ matrix is the same as for the with-interaction form.

Omitting the effects due to columns and interactions from a 2-way classification, e.g., equation (0), reduces it to a 1-way classification. The same is true of (1), where we would also omit the b_j's. However, this procedure cannot necessarily be satisfactorily extended to the estimation process. Deleting from $\mathbf{R}_z'\mathbf{R}_z$, the b columns and rows corresponding to the b_j's of the model (1) does not yield the estimation process for the 1-way classification of rows, with covariate. This is because in $\mathbf{R}_z'\mathbf{R}_z$ of (6) the first column of $\mathbf{Z}$ has been deleted to overcome the otherwise singularity of $\mathbf{R}_z'\mathbf{R}_z$, i.e., to overcome the fact that $\mathbf{Z}$ of, for example, (3) does not have full column rank. On the other hand, deleting from $\mathbf{R}_z'\mathbf{R}_z$ the (a - 1) columns and rows corresponding to the a_i's $i = 2, \cdots, a$ of model (1) does lead to the correct estimation process of the 1-way classification of columns, with covariate.

References

Searle, S. R. [1971]. Linear Models, Wiley, New York.

Zinger, Alexis [1974]. Personal Communication.