Summary

It is the purpose of this bulletin to show by elementary algebra that, for the randomized complete block experiment, the analysis of covariance of $Y$ using $X$ as a covariate is equivalent to the analysis of covariance of $Y-X$, using $X$ as a covariate.

Introduction

The user of statistics sometimes raises the question of whether to use $Y$ or $Y-X$ as a dependent variable with $X$ as the independent variable in a covariance analysis. Thus in an animal experiment where initial and final weights are observed, the experimenter may raise the question of whether final weight or gain in weight has more merit as a dependent variable when initial weight is to be the independent variable.

Notation

Let $X_{ij} =$ the observation on the variable $X$ for the $i$-th block and $j$-th treatment, $X_i. =$ the sum of the observations in the $i$-th block, $X_{.j} =$ the sum of the observations receiving the $j$-th treatment. The letter $x$ with subscripts is used to denote a deviation from the appropriate mean. The same notation is used for the variable $Y$.

Analyses of variance

I. $Y =$ the dependent variable, $X =$ the independent variable.

<table>
<thead>
<tr>
<th>Source</th>
<th>d.f.</th>
<th>$\Sigma x^2$</th>
<th>$\Sigma x y$</th>
<th>$\Sigma y^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Blocks</td>
<td>$b-1$</td>
<td>$\Sigma x_{i.}^2$</td>
<td>$\frac{\Sigma x_{i.} y_{i.}}{t}$</td>
<td>$\frac{\Sigma y_{i.}^2}{t}$</td>
</tr>
<tr>
<td>Treatments</td>
<td>$t-1$</td>
<td>$\Sigma x_{.j}^2$</td>
<td>$\frac{\Sigma x_{.j} y_{.j}}{b}$</td>
<td>$\frac{\Sigma y_{.j}^2}{b}$</td>
</tr>
<tr>
<td>Residual</td>
<td>$(b-1)(t-1)$</td>
<td>←— by subtraction —→</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>$bt-1$</td>
<td>$\Sigma x_{ij}^2$</td>
<td>$\Sigma x_{ij} y_{ij}$</td>
<td>$\Sigma y_{ij}^2$</td>
</tr>
</tbody>
</table>
II. Z = Y - X = the dependent variable, X = the independent variable.

<table>
<thead>
<tr>
<th>Source</th>
<th>d.f.</th>
<th>$\Sigma x_i^2$</th>
<th>$\Sigma x_i y_i$</th>
<th>$\Sigma z_i^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Blocks</td>
<td>b-1</td>
<td>$\frac{\Sigma x_i^2}{t}$</td>
<td>$\frac{\Sigma x_i y_i}{t}$</td>
<td>$\frac{\Sigma z_i^2}{t}$</td>
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<td>$\Sigma z_i^2$</td>
</tr>
</tbody>
</table>

Reduced sums of squares

\[ I. \quad \Sigma y_{ij}^2 - \frac{(\Sigma x_{ij} y_{ij})^2}{\Sigma x_{ij}^2} \]
\[ II. \quad \left\{ \Sigma y_{ij}^2 + \Sigma x_{ij}^2 - 2 \Sigma x_{ij} y_{ij} \right\} - \frac{(\Sigma x_{ij} y_{ij}^2 - \Sigma x_{ij}^2)^2}{\Sigma x_{ij}^2} \]
\[ = \left\{ \Sigma y_{ij}^2 - \frac{(\Sigma x_{ij} y_{ij})^2}{\Sigma x_{ij}^2} \right\} + \left\{ \frac{\Sigma x_{ij}^2 - (\Sigma x_{ij}^2)^2}{\Sigma x_{ij}^2} \right\} \]
\[ - 2 \left\{ \Sigma x_{ij} y_{ij} - \frac{(\Sigma x_{ij} y_{ij}) (\Sigma x_{ij}^2)}{\Sigma x_{ij}^2} \right\} \]

= Reduced Total sum of squares for I since each of the last two \{\ldots\} is zero.

Treatments, blocks, etc.

It is clear from above that any other classification in an n-way classification will have the same reduced sum of squares for I and II.

Treatments + Residual

\[ I. \quad \Sigma y_{ij}^2 - \frac{\Sigma y_{ij}^2}{t} - \frac{\left\{ \Sigma x_{ij} y_{ij} - \frac{\Sigma x_{ij} y_{ij}}{t} \right\}^2}{\Sigma x_{ij}^2 - \frac{\Sigma x_{ij}^2}{t}} \]

- 2
\[ \text{Reduced Treatment + Residual sum of squares for I since the extra terms are seen to add to zero.} \]

Residual

\[ \begin{align*}
\text{I.} & \quad \sum y_{ij}^2 - \frac{\sum y_{ij}^2}{t} - \frac{\sum y_{i.}^2}{b} - \left( \frac{\sum x_{ij} y_{ij} - \frac{\sum x_{ij} y_{i.}}{t} - \frac{\sum x_{ij} y_{.i}}{b}}{\sum x_{ij}^2 - \frac{\sum x_{ij}^2}{t} - \frac{\sum x_{ij}^2}{b}} \right)^2 \\
\text{II.} & \quad \sum y_{ij}^2 + \sum x_{ij}^2 - 2\sum x_{ij} y_{ij} - \frac{\sum y_{i.}^2 + \sum x_{i.}^2 - 2\sum x_{i.} y_{i.}}{t} \\
& \quad - \frac{\sum y_{.i}^2 + \sum x_{.i}^2 - 2\sum x_{.i} y_{.i}}{b}
\end{align*} \]
\[ \sum y_{ij}^2 - \frac{\sum y_{ij}}{b} \cdot \sum x_{ij}^2 \] 
\[ + \left[ \sum x_{ij}^2 - \frac{\sum x_{ij}}{b} \right] \] 
\[ = \sum x_{ij} \left( \frac{\sum x_{ij}}{b} - \frac{\sum x_{ij}}{t} \right)^2 \] 
\[ = \sum x_{ij} \left( \frac{\sum x_{ij}}{t} - \frac{\sum x_{ij}}{b} \right) \] 
\[ = \text{Reduced Residual sum of squares for I since the extra terms are seen to add to zero.} \]

Conclusions

Since all reduced sums of squares of interest are seen to be identical for I and II, the usual tests of adjusted treatment means and of the difference between regressions for lot means and within lots are seen to be identical.