Donna

Abstract for BU-533-M.

A hand-out for a talk on restrictions on models and constraints on solutions to normal equations when fitting linear models.

RESTRICTIONS ON MODELS AND CONSTRAINTS ON SOLUTIONS IN ANALYSIS OF VARIANCE*

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S. R. Searle

Biometrics Unit, Cornell University, Ithaca, N. Y.

Introductory Example

Suppose for the model $y_{ij} = \mu + \alpha_i + e_{ij}$, i=1,2,3 and j=1,2,3 that the normal equations are

$$\begin{aligned}
6\hat{\mu} + 2\hat{\alpha}_1 + 2\hat{\alpha}_2 + 2\hat{\alpha}_3 &= 476 \\
2\hat{\mu} + 2\hat{\alpha}_1 &= 142 \\
2\hat{\mu} + 2\hat{\alpha}_2 &= 156 \\
2\hat{\mu} + 2\hat{\alpha}_3 &= 178
\end{aligned} \qquad (1)$$

Sometimes these equations are solved by using the additional equation

$$\hat{\alpha}_1 + \hat{\alpha}_2 + \hat{\alpha}_3 = 0 \ . \qquad (2)$$

This is called a constraint on the solutions.

Comparable to (2) is an equation in parameters of the model,

$$\alpha_1 + \alpha_2 + \alpha_3 = 0 \ , \qquad (3)$$

which is called a restriction on the model.

The effect on estimation and on hypothesis testing of generalizations of (2) and (3) are now considered, especially for normal equations more general than those in (1), in particular for those that arise from unbalanced (unequal subclass numbered) data, of which the following is a simple example.

$$\begin{aligned}
7\mu^\circ + 3\alpha_1^\circ + 2\alpha_2^\circ + 2\alpha_3^\circ &= 553 \\
3\mu^\circ + 3\alpha_1^\circ &= 249 \\
2\mu^\circ + 2\alpha_2^\circ &= 156 \\
2\mu^\circ + 2\alpha_3^\circ &= 178
\end{aligned} \qquad (4)$$

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Summary of general linear model theory

Model: $\underset{\sim}{y} = \underset{\sim}{X}\underset{\sim}{b} + \underset{\sim}{e}$, with $\underset{\sim}{e} \sim N(\underset{\sim}{o}, \sigma^2 \underset{\sim}{I}_N)$. Call this the unrestricted model.

Normal equations: $\underset{\sim}{X}'\underset{\sim}{X}\underset{\sim}{b} = \underset{\sim}{X}'\underset{\sim}{y}$.

Solution: $\underset{\sim}{b}^{o} = \underset{\sim}{G}\underset{\sim}{X}'\underset{\sim}{y}$ with $\underset{\sim}{X}'\underset{\sim}{X}\underset{\sim}{G}\underset{\sim}{X}'\underset{\sim}{X} = \underset{\sim}{X}'\underset{\sim}{X}$.

Estimable functions: $\underset{\sim}{k}'\underset{\sim}{b}$ with $\underset{\sim}{k}' = \underset{\sim}{t}'\underset{\sim}{X}$ for some $\underset{\sim}{t}$.

B.L.U.E. (of estimable function): $\widehat{\underset{\sim}{k}'\underset{\sim}{b}} = \underset{\sim}{k}'\underset{\sim}{b}^{o}$, invariant to $\underset{\sim}{G}$ and $\underset{\sim}{b}^{o}$.

Example: In $y_{ij} = \mu + \alpha_i + e_{ij}$, $\alpha_i - \alpha_k$ is estimable with $\widehat{\alpha_i - \alpha_k} = \alpha_i^{o} - \alpha_k^{o}$.

Sums of squares: $R(\underset{\sim}{b}) = \underset{\sim}{y}'\underset{\sim}{X}\underset{\sim}{G}\underset{\sim}{X}'\underset{\sim}{y} = \underset{\sim}{b}^{o\prime}\underset{\sim}{X}'\underset{\sim}{y}$, invariant to $\underset{\sim}{G}$ and $\underset{\sim}{b}^{o}$.

$SSE = \underset{\sim}{y}'(\underset{\sim}{I} - \underset{\sim}{X}\underset{\sim}{G}\underset{\sim}{X}')\underset{\sim}{y}$.

$\hat{\sigma}^2 = SSE/[N - r(\underset{\sim}{X})]$, $r(\underset{\sim}{X}) \equiv$ rank of $\underset{\sim}{X}$.

Hypothesis testing:

$H : \underset{\sim}{K}'\underset{\sim}{b} = \underset{\sim}{m}$ with $\underset{\sim}{K}'\underset{\sim}{b}$ estimable and $\underset{\sim}{K}'$ of full row rank.

$$F(H) = \frac{(\underset{\sim}{K}'\underset{\sim}{b}^{o} - \underset{\sim}{m})'(\underset{\sim}{K}'\underset{\sim}{G}\underset{\sim}{K})^{-1}(\underset{\sim}{K}'\underset{\sim}{b}^{o} - \underset{\sim}{m})}{\hat{\sigma}^2 r(\underset{\sim}{K})}$$

$$\sim F_{r(\underset{\sim}{K}),\, N-r(\underset{\sim}{X})} \quad \text{under } H.$$

$$\underset{\sim}{b}_H^{o} = \underset{\sim}{b}^{o} - \underset{\sim}{G}\underset{\sim}{K}(\underset{\sim}{K}'\underset{\sim}{G}\underset{\sim}{K})^{-1}(\underset{\sim}{K}'\underset{\sim}{b}^{o} - \underset{\sim}{m}).$$

Restricted models: restrictions on parameters

Restrictions: $\underset{\sim}{P}'\underset{\sim}{b} = \underset{\sim}{\delta}$, with $\underset{\sim}{P}'$ full row rank and $\underset{\sim}{P}$ and $\underset{\sim}{\delta}$ known.

Model: $\underset{\sim}{y} = \underset{\sim}{X}\underset{\sim}{b} + \underset{\sim}{e}$ and $\underset{\sim}{P}'\underset{\sim}{b} = \underset{\sim}{\delta}$.

Normal equations: $\underset{\sim}{X}'\underset{\sim}{X}\underset{\sim}{b}_r^o + \underset{\sim}{P}\underset{\sim}{\theta} = \underset{\sim}{X}'\underset{\sim}{y}$, $\underset{\sim}{\theta}$ being a vector of Lagrange multipliers.

$\underset{\sim}{P}'\underset{\sim}{b}_r^o = \underset{\sim}{\delta}$

Subscript r on $\underset{\sim}{b}_r^o$ to distinguish restricted model.

Solutions: 2 cases, (a) $\underset{\sim}{P}'\underset{\sim}{b}$ not estimable and (b) $\underset{\sim}{P}'\underset{\sim}{b}$ estimable.

(a) $\underset{\sim}{P}'\underset{\sim}{b}$ not estimable in $\underset{\sim}{y} = \underset{\sim}{X}\underset{\sim}{b} + \underset{\sim}{e}$ with $\underset{\sim}{P}'\underset{\sim}{b} = \underset{\sim}{\delta}$

Solution: $\underset{\sim}{b}_r^o = \underset{\sim}{G}\underset{\sim}{X}'\underset{\sim}{y} + (\underset{\sim}{G}\underset{\sim}{X}'\underset{\sim}{X} - \underset{\sim}{I})\underset{\sim}{z}$ for $\underset{\sim}{z}$ such that $\underset{\sim}{P}'(\underset{\sim}{G}\underset{\sim}{X}'\underset{\sim}{X} - \underset{\sim}{I})\underset{\sim}{z} = \underset{\sim}{\delta} - \underset{\sim}{P}'\underset{\sim}{G}'\underset{\sim}{X}'\underset{\sim}{y}$

= one of the solutions to $\underset{\sim}{X}'\underset{\sim}{X}\underset{\sim}{b} = \underset{\sim}{X}'\underset{\sim}{y}$

= one of the solutions in the unrestricted model.

Sums of squares: Same as in unrestricted model.

Estimable functions: Suppose $\underset{\sim}{k}'\underset{\sim}{b}$ is estimable in the unrestricted model.

Then $(\underset{\sim}{k}' + \underset{\sim}{\lambda}'\underset{\sim}{P})\underset{\sim}{b}$ for $\underset{\sim}{\lambda}'\underset{\sim}{\delta} = 0$, is estimable in restricted model. (Note: $\underset{\sim}{\delta}$ is often null.)

Although $(\underset{\sim}{k}' + \underset{\sim}{\lambda}'\underset{\sim}{P})\underset{\sim}{b}$ is estimable in the restricted model, in general it is not estimable in the unrestricted model because $\underset{\sim}{P}'\underset{\sim}{b}$ is not.

Examples

Unrestricted Model ($\Sigma\alpha_i$ and $\Sigma n_i\alpha_i$ are not estimable)	Restricted Model #1 $\Sigma\alpha_i = 0$	Restricted Model #2 $\Sigma n_i\alpha_i = 0$
$\widehat{\alpha_1 - \frac{1}{3}(\alpha_1 + \alpha_2 + \alpha_3)} = \bar{y}_{1.} - \bar{y}_{..}$	$\hat{\alpha}_1 = \bar{y}_{1.} - \bar{y}_{..}$	
$\widehat{\mu + \frac{1}{3}\Sigma\alpha_i} = \frac{1}{3}\Sigma\bar{y}_{i.}$	$\hat{\mu} = \frac{1}{3}\Sigma\bar{y}_{i.}$	
$\widehat{\mu + \Sigma n_i\alpha_i/n_.} = \bar{y}_{..}$		$\hat{\mu} = \bar{y}_{..}$

Hypothesis testing:

Suppose $H:\underset{\sim}{K}'\underset{\sim}{b} = \underset{\sim}{m}$ is testable in the unrestricted model.

Then $H:(\underset{\sim}{K}' + \underset{\sim}{L}\underset{\sim}{P}')\underset{\sim}{b} = \underset{\sim}{m} + \underset{\sim}{L}\underset{\sim}{\delta}$ is testable in the restricted model.

$F(H)$ and $\underset{\sim}{b}_H^o$ have same form as in the unrestricted model.

(b) $P'b$ estimable in $y = Xb + e$ with $P'b = \delta$

Solution: $b_r^o = b^o - GP(P'GP)^{-1}(P'b^o - \delta)$

Sum of squares: $SSE_r = SSE + (P'b^o - \delta)(P'GP)^{-1}(P'b^o - \delta)$

$$\hat{\sigma}_r^2 = \frac{SSE_r}{N - r(X) - r(P')} > \hat{\sigma}^2 = \frac{SSE}{N - r(X)} .$$

Estimable functions: Suppose $k'b$ is estimable in the unrestricted model.

Then $(k' + \lambda'P)b$ for $\lambda'\delta = 0$ is estimable in the restricted model.

This is the same form as when $P'b$ is not estimable.

But here $P'b$ is estimable in the unrestricted model and so $(k' + \lambda'P)b$ is just one of the estimable functions of the unrestricted model.

Hypothesis testing: $H_r: K'b = m$.

All hypotheses testable in the unrestricted model are testable in the restricted model, and vice versa.

$$SSE_{r,H} = SSE + (Q'b^o - \ell)(Q'GQ)^{-1}(Q'b^o - \ell)$$

$$\text{with } Q' = \begin{bmatrix} P' \\ K' \end{bmatrix} \text{ and } \ell' = \begin{bmatrix} \delta \\ m \end{bmatrix}$$

$$F(H_r) = \frac{SSE_{r,H} - SSE_r}{\hat{\sigma}_r^2 \, r(K)}$$

$$= \frac{(K'b_r^o - m)'[K'GK - K'GP(P'GP)^{-1}P'GK]^{-1}(K'b_r^o - m)}{\hat{\sigma}_r^2 \, r(K)} .$$

[Timm and Carlson (1973)]

Constraints on solutions

None of the preceding general results demands constraints.

All we need is one solution $\tilde{b}^o$ to $\tilde{X}'\tilde{X}\tilde{b}^o = \tilde{X}'\tilde{y}$.

An easily-obtained solution often comes from using some particular constraint.

An easily-used constraint is: some b_i^o's = 0.

This implies deleting rows and columns of $\tilde{X}'\tilde{X}\tilde{b}^o = \tilde{X}'\tilde{y}$.

After doing so, suppose the equations so modified are

$$(\tilde{X}'\tilde{X})_m \tilde{b}_m^o = (\tilde{X}'\tilde{y})_m \text{ with solution } \tilde{b}_m^o = [(\tilde{X}'\tilde{X})_m]^{-1}(\tilde{X}'\tilde{y})_m \quad .$$

Then $\tilde{G}$ for $\tilde{X}'\tilde{X}\tilde{G}\tilde{X}'\tilde{X} = \tilde{X}'\tilde{X}$ is

$$\tilde{G}_* = \{\tilde{X}'\tilde{X} \text{ with } (\tilde{X}'\tilde{X})_m \text{ replaced by } [(\tilde{X}'\tilde{X})_m]^{-1}, \text{ and all else } 0\} \quad . \qquad (5)$$

Example For the constraint $\mu^o = 0$ in equation (4)

$$(\tilde{X}'\tilde{X})_m \tilde{b}_m^o = (\tilde{X}'\tilde{y})_m \text{ is } \begin{matrix} 3\alpha_1^o & & = 219 \\ & 2\alpha_1 & = 156 \\ & 2\alpha_2^o & = 178 \end{matrix} \text{ with } \tilde{G} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix} .$$

Constraints and Restrictions together

The use of either does not imply the other:

$$\text{e.g. } \tilde{C}'\tilde{b}^o = \tilde{\gamma} \not\Rightarrow \tilde{C}'\tilde{b} = \tilde{\gamma} \text{ and } \tilde{P}'\tilde{b} = \tilde{\delta} \not\Rightarrow \tilde{P}'\tilde{b}^o = \tilde{\delta} \quad .$$

Restricted model: $\tilde{y} = \tilde{X}\tilde{b} + \tilde{e}$, with $\tilde{P}'\tilde{b} = \tilde{\delta}$, $\tilde{P}'\tilde{b}$ estimable.

(With $\tilde{P}'\tilde{b}$ not estimable, solution $\tilde{b}_r^o$ is just a solution of $\tilde{X}'\tilde{X}\tilde{b}^o = \tilde{X}'\tilde{y}$; p. 3.)

Constraints: some set of b_i^o's = 0. (6)

Solutions: (i) satisfying $\tilde{X}'\tilde{X}\tilde{b}^o = \tilde{X}'\tilde{y}$

$$\tilde{b}_*^o = \tilde{G}_*\tilde{X}'\tilde{y} \text{ using } \tilde{G}_* \text{ from (5) using (6).}$$

(ii) also satisfying $\tilde{P}'\tilde{b}^o = \tilde{\delta}$

$$\tilde{b}_{r,*}^o = \tilde{b}_* + (\tilde{G}_*\tilde{X}'\tilde{X} - \tilde{I})\tilde{z}$$

for $\tilde{z}$ such that $\tilde{P}'(\tilde{G}_*\tilde{X}'\tilde{X} - \tilde{I})\tilde{z} = \tilde{\delta} - \tilde{P}'\tilde{G}_*\tilde{X}'\tilde{y}$.

Sums of squares: $SSE_r = SSE$ because $\tilde{P}'\tilde{b}$ is not estimable (p. 3).