THE INVERSE OF SOME CIRCULANT MATRICES

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Abstract

A method that is useful for inverting certain patterned matrices is described and applied to some circulant matrices that arise in statistics. The method is based on solving recurrence equations.

Method

The following method of inverting certain patterned matrices was brought to my attention in 1966 by the author of Kounias [1968].

Suppose $A$ is a patterned matrix of order $n$ whose inverse is sought. Consider the equations

$$
\begin{bmatrix}
z_1 \\
z_2 \\
\vdots \\
z_n
\end{bmatrix}
A
=
\begin{bmatrix}
w \\
w^2 \\
\vdots \\
w^n
\end{bmatrix}
$$

written in an obvious way as $Az = w$. Their solution for $z$ is

$$
z = A^{-1}w
$$

where, on writing
the $i$'th element of (2) is

$$z_i = a_{i1}w + a_{i2}w^2 + \ldots + a_{in}w^n \quad (3)$$

Hence the $ij$'th element of $A^{-1}$ is

$$a_{ij}^* = \text{coefficient of } w^j \text{ in } z_i \quad (4)$$

It is the pattern in $A$ that gives rise to equations (1) being recurrent equations for $z_i, z_{i-1}, \ldots$. Solving them by traditional methods of solving such equations gives an expression for $z_i$ corresponding to (3), which by (4) then yields elements of $A^{-1}$.

**An easy example**

Consider

$$A = aI + bJ$$

where $I$ is an identity matrix and $J$ is a square matrix having every element unity.

For order 3

$$A = \begin{bmatrix} a+b & b & b \\ b & a+b & b \\ b & b & a+b \end{bmatrix}$$

For order $n$ it is well known that
We derive the elements of $A^{-1}$ using (1), (3) and (4).

The nature of $A$, i.e., its pattern, is such that the $i$'th element of equation (1) is

$$az_i + b \sum_{i=1}^{n} z_i = w^i . \quad (6)$$

Summing over $i$ gives

$$a \Sigma z_i + nb \Sigma z_i = \Sigma w^i \quad \text{so that} \quad \Sigma z_i = \Sigma w^i/(a+nb) .$$

Substitution in (6) leads to

$$z_i = \frac{w^i}{a} - \frac{b}{a(a+nb)} \Sigma w^i . \quad (7)$$

This corresponds to (3), and so (4) gives

$$a^i_{ii} = \text{coefficient of } w^i \text{ in (7)} = \frac{1}{a} - \frac{1}{a(a+nb)}$$

and

$$a^i_{ij} = \text{coefficient of } w^j \text{ in (7)} = \frac{-1}{a(a+nb)} , \quad \text{for } i \neq j .$$

These are exactly the elements of (5).
A harder example

Before considering some circulant matrices we first find the inverse of a matrix dealt with in Kounias [1968]. A 5 \times 5 example is

\[
A = \begin{bmatrix}
a & b & 0 & 0 & 0 \\
c & a & b & 0 & 0 \\
0 & c & a & b & 0 \\
0 & 0 & c & a & b \\
0 & 0 & 0 & c & a
\end{bmatrix}
\]  

(8)

Equations \(Az = w\) of (1) give rise to

\[
cz_{i-1} + az_i + bz_{i+1} = w^i
\]

(9)

with boundary conditions from (8) being, for \(A\) of order \(n\),

\[
z_0 = 0 \quad \text{and} \quad z_{n+1} = 0
\]

(10)

Kounias [1968] gives the solution of (9) directly as

\[
z_i = \frac{i+1}{bw^2+aw+c} w^i + py_1^i + qy_2^i
\]  

(10a)

where \(y_1\) and \(y_2\) are the roots of \(by^2 + ay + c = 0\), and \(p\) and \(q\) are constants determined from the boundary conditions (10); i.e.,

\[
o = \frac{w}{bw^2+aw+c} + p + q
\]  

(10b)

and

\[
o = \frac{w^{n+2}}{bw^2+aw+c} + py_1^{n+1} + qy_2^{n+1}
\]  

(10c)
These are easily solved for \( p \) and \( q \) which are then substituted into (10a).

**Solving recurrence equations**

The details for deriving (10a) are instructive. First define

\[
  u_i = z_i / w^i
\]

and rewrite (9) as

\[
  (c/w)u_{i-1} + au_i + bwu_{i+1} = 1
\]

Recurrence equations of this nature are solved (see, for example Durell and Robson [1946], chapter 11) by first finding solutions to

\[
  bw^2 + ax + c/w = 0,
\]

i.e., to

\[
  bw^2 x^2 + axw + c = 0.
\]

Solutions are

\[
  x_1 = y_1 / w \quad \text{and} \quad x_2 = y_2 / w
\]

where

\[
  y_1 = \frac{-a + \sqrt{a^2 - 4bc}}{2b} \quad \text{and} \quad y_2 = \frac{-a - \sqrt{a^2 - 4bc}}{2b}.
\]

The solution to (12) is then

\[
  u_i = ax_1^i + bx_2^i + \gamma
\]

for \( \alpha, \beta \) and \( \gamma \) yet to be determined.

Initial conditions (10) are used first. From (10) and (11) \( u_0 = 0 \), and with

\[ i = 0 \text{ in (16)} \]
Hence (16) is

\[ u_1 = \alpha(x_1^i - 1) + \beta(x_2^i - 1) \]  

(17)

From (16) and (12) \( u_{n+1} = 0 \) also, and with \( i = n + 1 \) in (17)

\[ u_{n+1} = 0 = \alpha(x_1^{n+1} - 1) + \beta(x_2^{n+1} - 1) \]

which on defining

\[ k = \frac{x_1^{n+1} - 1}{x_2^{n+1} - 1} \]  

(18)

gives \( \beta = -kx \) and so (17) becomes

\[ u_1 = \alpha[x_1^i - 1 - k(x_2^i - 1)] , \]
\[ u_1 = \alpha(k - 1 + x_1^i - kx_2^i) , \]

and so on using (11)

\[ z_1 = w^i u_1 = w^i \alpha(k - 1 + x_1^i - kx_2^i) \]  

(19)

To determine \( \alpha \), use \( i = 1 \) in (9) and with (10) get

\[ az_1 + bz_2 = w \]  

(20)

and in this substitute for \( z_1 \) and \( z_2 \) from (19). Thus

\[ a\alpha(k-1+x_1-kx_2) + bw^2\alpha(k-1+x_2^2-kx_2^2) = w \]

or

\[ \alpha[(a+bw)(k-1) + bwx_1^2 + ax_1 - k(bwx_2^2+ax_2^2)] = 1 \]
But since $x_1$ and $x_2$ satisfy (13) this becomes

$$\alpha[(a+bw)(k-1) - c/w - k(-c/w)] = 1,$$

giving

$$\alpha = \frac{1}{(k-1)(a+bw+c/w)} = \frac{w}{(k-1)(bw^2+aw+c)}.$$  \hspace{1cm} (21)

Hence in (19)

$$z_i = \frac{w^i+1}{bw^2+aw+c} \left[ 1 + \frac{x_1 - kx_2}{k-1} \right].$$  \hspace{1cm} (22)

On substituting for $k$ from (18) and for $x_1$ and $x_2$ from (14) it will be found that the solution (22) is identical to (10a) with $p$ and $q$ satisfying (10b).

Elements of $A^{-1}$

To find $a_{ij}$, the $(i,j)$'th element of $A^{-1}$, we need as in (4), the coefficient of $w^j$ in $z_i$. To derive this we rearrange (22) as a polynomial in $w$, using the fact that in (14) and (18) $x_1$, $x_2$, and $k$ are functions of $w$. First we express $1/(bw^2+aw+c)$ as a power series in $w$, by noting that $x = 1$ in (14) gives

$$bw^2 + aw + c = 0.$$

Hence, for $y_1$ and $y_2$ of (15)

$$bw^2 + aw + c = b(w-y_1)(w-y_2)$$

so that

$$\frac{1}{bw^2 + aw + c} = \frac{1}{b} \left( \frac{r}{w-y_1} + \frac{s}{w-y_2} \right).$$
where \( r \) and \( s \) are determined by

\[
\begin{align*}
  r + s &= 0 \\
  ry_2 + sy_1 &= -1 ;
\end{align*}
\]

i.e.,

\[
  r = -s = 1/(y_1-y_2) .
\]

Hence

\[
\frac{1}{bw^2 + aw + c} = \frac{1}{b(y_1-y_2)} \left( \frac{1}{w-y_1} - \frac{1}{w-y_2} \right)
\]

\[
= \frac{1}{b(y_1-y_2)} \left[ \frac{1}{w} \sum_{k=0}^{\infty} \left( \frac{y_1}{w} \right)^k - \frac{1}{w} \sum_{k=0}^{\infty} \left( \frac{y_2}{w} \right)^k \right]
\]

\[
= \frac{1}{bw(y_1-y_2)} \sum_{k=1}^{\infty} \frac{y_1^k - y_2^k}{w^k}, \tag{23}
\]

the first term in the summation being for \( k = 1 \), because \( y_1^0 - y_2^0 = 1 - 1 = 0 \).

Next, from (18) and (14) we have

\[
\frac{1}{k-1} = \frac{y_{n+1}^k - w_{n+1}^k}{y_{n+1}^k - y_{n+1}^2} \quad \text{and} \quad \frac{k}{k-1} = \frac{y_{n+1}^k - w_{n+1}^k}{y_{n+1}^k - y_{n+1}^2}
\]

so that substituting these and (23) into (22) gives

\[
z_i = \frac{1}{bw(y_1-y_2)} \left[ \sum_{k=1}^{\infty} \frac{y_1^k - y_2^k}{w} \left[ \frac{y_1(y_{n+1}^k-w_{n+1}^k) - y_2(y_{n+1}^k-w_{n+1}^2)}{w(y_{n+1}^k-y_{n+1}^2)} \right] + \frac{y_1 y_{n+1}^k - y_2 y_{n+1}^k}{w y_{n+1}^k - y_{n+1}^2} \sum_{k=1}^{\infty} (y_1 y_2 w^{-k} - y_1 y_2 y_{n+1}^k) \right] \tag{24}
\]
Since our sole use for $z_1$ is that of finding the coefficients of powers of $w$ in $z_1$, for positive powers only, in fact only for $w, w^2, \ldots, w^k$, we can exclude from (24) all powers that are less than 1 or greater than $n$. Hence we take

$$z'_i = \frac{1}{b(y_1-y_2)} \left[ \sum_{k=1}^{i-1} (y_1^{k} - y_2^{k}) w^{i-k} - \frac{y_1^i - y_2^i}{y_1^{n+1} - y_2^{n+1}} \sum_{k=1}^{n} (y_1^{k} - y_2^{k}) w^{n+1-k} \right].$$

Then, on putting $j = i-k = 1, 2, \ldots, i-1$ in the first summation and $j = n+1-k = 1, 2, \ldots, n$ in the second we have:

$$z'_i = \frac{1}{y(y_1-y_2)} \left[ \sum_{j=1}^{i-1} (y_1^{i-j} - y_2^{i-j}) w^j - \frac{y_1^i - y_2^i}{y_1^{n+1} - y_2^{n+1}} \sum_{j=1}^{n} (y_1^{n+1-j} - y_2^{n+1-j}) w^j \right].$$

The $ij$'th element $a^*_{ij}$ of $A^{-1}$ is the coefficient of $w^j$ in $z_i$. Because the two summations in $z'_i$ have different upper limits there will be two forms of $a^*_{ij}$:

$$j < i : \quad a^*_{ij} = \frac{1}{b(y_1-y_2)} \left[ y_1^{i-j} - y_2^{i-j} - \frac{y_1^i - y_2^i}{y_1^{n+1} - y_2^{n+1}} (y_1^{n+1-j} - y_2^{n+1-j}) \right]$$

$$= \frac{-y_1^{i-j} y_1^{n+1} y_2^{i-j} + y_1^{i} y_2^{n+1-j} + y_1^{i-j} y_2^{n+1-j}}{b(y_1-y_2)(y_1^{n+1} - y_2^{n+1})}$$

$$= \frac{-(y_1^{i-j} y_2^{i}) (y_1 - y_2)(y_1^{n+1} - y_2^{n+1})}{b(y_1-y_2)(y_1^{n+1} - y_2^{n+1})}$$

$$= \frac{-(y_1^{i-j} y_2^{i})(y_1^{n+1-i} y_2^{n+1-i})}{b(y_1-y_2)(y_1^{n+1} - y_2^{n+1})}$$

and on noting that $y_1 y_2 = c/b$ this reduces to

$$j < i : \quad a^*_{ij} = \frac{c^{i-j} (y_1^{j} - y_2^{j})(y_1^{n+1-i} - y_2^{n+1-i})}{(y_1 - y_2)(y_1^{n+1} - y_2^{n+1})}.$$
These results are slightly simplified forms of those given in Kounias [1968]. Alternative forms that display their structure a little more are based on defining

\[ \Delta_t = y_1^t - y_2^t. \]

Then for

\[
a_{ij}^* = \begin{cases} 
\frac{\Delta_i \Delta_{n+1}^i}{b_1^i-j+1} & \text{if } j \geq i \\
\frac{-\Delta_i \Delta_{n+1}^i}{b_1^i-j+1} & \text{if } j < i
\end{cases}
\]

Although \( \Delta_1 = y_1 - y_2 \) occurs in the denominators of these expressions it is also a factor of the numerators. Indeed, \( \Delta_1^2 \) is a factor of both numerators and denominators. When \( y_1 = y_2 \) this factor needs to be removed prior to using the formulae.

**Example**

\[
a = -3 \quad \begin{bmatrix} -3 & 1 & 0 & 0 \\
b = 1 & 2 & -3 & 1 \\
c = 2 & 0 & 2 & -3 \\
n = 4 & 0 & 0 & 2 & 3
\end{bmatrix}
\]

\[ \sqrt{a^2 - 4bc} = \sqrt{9-8} = 1 \]
\[ y_1 = \frac{3+1}{2} = 2 \quad y_2 = \frac{3-1}{2} = 1 \]

\[ t : 1 \quad 2 \quad 3 \quad 4 \quad 5 \]
\[ \Delta_t = 2^t - 1^t : 1 \quad 3 \quad 7 \quad 15 \quad 31 \]
\[ b \Delta_t a_{n+1} = 31 \]

\[
\begin{bmatrix}
15 & 7 & 3 & 1 \\
14 & 21 & 9 & 3 \\
12 & 18 & 21 & 7 \\
8 & 12 & 14 & 15
\end{bmatrix}
\]

It is easily verified that \( AA^{-1} = I \).

**Some Circulant Matrices**

Certain properties of circulant matrices, e.g., matrices of the form

\[
\begin{bmatrix}
    a & b & c & d & e \\
    e & a & b & c & d \\
    d & e & a & b & c \\
    c & d & e & a & b \\
    b & c & d & e & a
\end{bmatrix}
\]

are well known. In particular, the inverse of a circulant matrix is a circulant matrix although in general, the form of the elements is not known. Special classes of circulant matrices arising in design of experiment problems considered by Anderson [1972] can, however, be inverted by the preceding method.
Case 1

The following $5 \times 5$ example illustrates the general form:

$$A = \begin{bmatrix}
a & b & 0 & 0 & c \\
c & a & b & 0 & 0 \\
0 & c & a & b & 0 \\
0 & 0 & c & a & b \\
b & 0 & 0 & c & a \\
\end{bmatrix} \quad (25)$$

This is simply (8) with non-zero elements in the upper right and lower left corners.

Equations $Az = w$ of (1) for inverting $A$ are

$$cz_i + az_i + bz_{i+1} = w^i$$

the same as (9). But now, from (25), the boundary conditions instead of (10) are,

for $A$ of order $n$,

$$z_0 = z_n \quad \text{and} \quad z_{n+1} = z_1 \quad (26)$$

The solution is therefore as in (10a),

$$z_i = \frac{w^{i+1}}{bw^2 + aw + c} + py_1^i + qy_2^i \quad (27)$$

with $p$ and $q$ to be determined from (26). For simplicity write

$$d = bw^2 + aw + c \quad .$$

With $i = 0$ and $i = n$ in (27), (26) then gives

$$w/d + p + q = w^{n+1}/d + py_1^n + qy_2^n \quad (28)$$
and \( i = 1 \) and \( i = n+1 \) gives

\[
\frac{w^2}{d} + py_1 + cy_2 = w^{n+2}/d + py_1^{n+1} + qy_2^{n+2}.
\]

(29)

These can be rewritten as

\[
\begin{bmatrix}
1 & 1 \\
y_1 & y_2 \\
\end{bmatrix}
\begin{bmatrix}
(1-y_1^n) \\
(1-y_2^n) \\
\end{bmatrix}
= \frac{-w(1-w^n)}{d} \begin{bmatrix}
1 \\
w \\
\end{bmatrix}
\]

(30)

so that

\[
\begin{bmatrix}
(1-y_1^n) & (1-y_1^n) \\
(1-y_2^n) & (1-y_2^n) \\
\end{bmatrix}
= \frac{-w(1-w^n)}{d(y_2-y_1)} \begin{bmatrix}
y_2 & -1 \\
y_1 & 1 \\
\end{bmatrix} \begin{bmatrix}
1 \\
w \\
\end{bmatrix}
\]

giving

\[
p = \frac{w(1-w^n)(y_2-w)}{d(y_1-y_2)(1-y_1^n)}
\]

(31)

and

\[
q = \frac{-w(1-w^n)(y_1-w)}{d(y_1-y_2)(1-y_2^n)}. \tag{32}
\]

Hence in (10a)

\[
z_i = \frac{w+1}{bw^2 + aw + c} + \frac{w(1-w^n)}{(bw^2 + aw + c)(y_1-y_2^n)} \left[ \frac{(y_2-w)y_1^i}{1-y_1^n} - \frac{(y_1-w)y_2^i}{1-y_2^n} \right] - \frac{(y_1-y_2^n)(y_1^i-y_2^i)}{(1-y_1^n)(1-y_2^n)}.
\]

(33)

Then, using (23)
Now our sole interest is in positive powers of w: w, w², ..., wⁿ. Hence in the first term we can put \( j = 1-k \) for \( j = 1, \ldots, i-1 \) and so have

\[
\sum_{j=1}^{i-1} (y_1^{i-j} - y_2^{i-j}) w^j.
\]

And in the second term the \( w^{-k} \) can be ignored, and we can let \( j = n-k \), for \( j = 1, \ldots, n-1 \), so getting

\[
\sum_{j=1}^{n-1} (y_1^{n-j} - y_2^{n-j}) w^j.
\]

Similarly in the third term the \( w^{1-k} \) can be ignored, and the second part for \( j = n+1-k \) becomes

\[
\sum_{j=1}^{n} (y_1^{n+1-j} - y_2^{n+1-j}) w^j.
\]

In this way \( z_1 \) of (34) becomes
\[
\begin{aligned}
  z_i' &= \frac{1}{b(y_1 - y_2)^2} \left[ (y_1 - y_2) \sum_{j=1}^{i-1} (y_1^{i-j} - y_2^{i-j}) w^j \right. \\
  &\quad \left. - \left( \frac{y_2^i}{1-y_2^n} - \frac{y_1^i}{1-y_1^n} \right) \sum_{j=1}^{n-1} (y_1^{n-j} - y_2^{n-j}) w^j \right] \\
  &\quad + \left( \frac{y_1^i}{1-y_1^n} - \frac{y_2^i}{1-y_2^n} \right) \sum_{j=1}^{n} (y_1^{n+1-j} - y_2^{n+1-j}) w^j \right).
\end{aligned}
\]

Then for
\[
A^{-1} = \{ a_{ij} \}, \quad a_{ij}^* = \text{coefficient of } w^j \text{ in } z_i'
\]

and because of the different upper limits in the summations there are three cases:

1) \( i < n \):
\[
a_{ij}^* = \frac{1}{b(y_1 - y_2)^2} \left[ (y_1 - y_2)(y_1^{i-j} - y_2^{i-j}) - \left( \frac{y_2^i}{1-y_2^n} - \frac{y_1^i}{1-y_1^n} \right) (y_1^{n-j} - y_2^{n-j}) \right. \\
  &\quad \left. + \left( \frac{y_1^i}{1-y_1^n} - \frac{y_2^i}{1-y_2^n} \right) (y_1^{n+1-j} - y_2^{n+1-j}) \right].
\]

2) \( n \geq j \geq i \):
\[
a_{ij}^* = \frac{1}{b(y_1 - y_2)^2} \left[ (y_2^i - y_1^i) (y_1^{n-j} - y_2^{n-j}) + \left( \frac{y_1^i}{1-y_1^n} - \frac{y_2^i}{1-y_2^n} \right) (y_1^{n+1-j} - y_2^{n+1-j}) \right].
\]

3) \( n = j \geq i \):
\[
a_{ij}^* = \frac{1}{b(y_1 - y_2)} \left( \frac{y_1^i}{1-y_1^n} - \frac{y_2^i}{1-y_2^n} \right).
\]

In the first two of these the term in \( 1/(1-y_1^n) \) is
\[
\frac{-y_2^i y_1^{j-n} + y_1^{i+n-j} y_2^j + y_1^{i+n+1-j} - y_1^{i+n+1-j} y_2^j}{1-y_1^n} = \frac{y_1^{n+1-j} (y_1 - y_2)}{1-y_1^n}
\]
and the term in \( \frac{1}{1-y_2^n} \) is

\[
\frac{y_1^{1+n-j} - y_2^{1+n-j} - y_1^{1+n-j} + y_2^{1+n-j}}{1-y_2^n} \left( \frac{y_2^{i-j} - y_1^{i-j} + y_1^{n+i-j} - y_2^{n+i-j}}{1-y_2^n} \right)
\]

so that the three classes of elements of \( A^{-1} \) are

\[
\begin{align*}
C_1, j < i: & \quad s_{ij}^* = \frac{1}{b(y_1-y_2)} \left( y_1^{1-j} - y_2^{1-j} + y_1^{n+i-j} - y_2^{n+i-j} \right) \\
& = \frac{1}{b(y_1-y_2)} \left( \frac{y_1^{i-j}}{1-y_1^n} - \frac{y_2^{i-j}}{1-y_2^n} \right) \\
C_2, n > j \geq i: & \quad s_{ij}^* = \frac{1}{b(y_1-y_2)} \left( y_1^{n+i-j} - y_2^{n+i-j} \right) \\
C_3, n = j \geq i: & \quad s_{in}^* = \frac{1}{b(y_1-y_2)} \left( \frac{y_1^i}{1-y_1^n} - \frac{y_2^i}{1-y_2^n} \right)
\end{align*}
\]

The only difference between these three classes is the powers of \( y_1 \) and \( y_2 \) involved in the numerator of each.

<table>
<thead>
<tr>
<th>Class</th>
<th>Powers of ( y_1 ) and ( y_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C_1 )</td>
<td>( i-j )</td>
</tr>
<tr>
<td>( C_2 )</td>
<td>( n+i-j )</td>
</tr>
<tr>
<td>( C_3 )</td>
<td>( i )</td>
</tr>
</tbody>
</table>

As has already been stated, the inverse of the circulant matrix \( A \) is also a circulant matrix, meaning that
\[ a_{i+1,j+1}^* = a_{i,j}^* \quad \text{for} \quad i,j = 1, 2, \ldots, n \] (35)

where \( n+1 \) occurring as a subscript is replaced by 1; e.g.,

\[ a_{n+1,j}^* \overset{\text{def}}{=} a_{1,j}^* \quad \text{and} \quad a_{i,n+1}^* \overset{\text{def}}{=} a_{i,1}^*. \] (36)

For \( C_1 \) and \( C_2 \) it is clear that so long as \( a_{i+1,j+1}^* \) is in the same class as \( a_{i,j}^* \) then (35) holds true. This is clearly so because the powers of \( y_1 \) and \( y_2 \) in classes \( C_1 \) and \( C_2 \) are \( i-j = i+1 - (j+1) \) and \( n+i-j = n+i+1 - (j+1) \). The situations when \( a_{i+1,j+1}^* \) is in a different class from \( a_{i,j}^* \) are shown below.

<table>
<thead>
<tr>
<th>Class</th>
<th>Sub-class</th>
<th>Element</th>
<th>Power of y's</th>
<th>Element</th>
<th>Class</th>
<th>Power of y's</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C_1 )</td>
<td>( \overset{\text{n},j&lt;n-1}{=} )</td>
<td>( a_{n,j}^* )</td>
<td>( n-j )</td>
<td>( a_{n+1,j+1}^* = a_{1,j+1} )</td>
<td>( C_2 )</td>
<td>( n+1-(j+1) = n-j )</td>
</tr>
<tr>
<td>( C_1 )</td>
<td>( \overset{\text{n},j=n-1}{=} )</td>
<td>( a_{n,n-1}^* )</td>
<td>( n-(n-1) = 1 )</td>
<td>( a_{n+1,n} = a_{1,n} )</td>
<td>( C_3 )</td>
<td>( 1 )</td>
</tr>
<tr>
<td>( C_2 )</td>
<td>( j=n-1 )</td>
<td>( a_{i,n-1}^* )</td>
<td>( n+i-(n-1) = i+1 )</td>
<td>( a_{i+1,n} )</td>
<td>( C_3 )</td>
<td>( i+1 )</td>
</tr>
<tr>
<td>( C_3 )</td>
<td>( i&lt;n )</td>
<td>( a_{i,n}^* )</td>
<td>( i )</td>
<td>( a_{i+1,n+1} = a_{i+1,1} )</td>
<td>( C_1 )</td>
<td>( i+1-1 = i )</td>
</tr>
<tr>
<td>( C_3 )</td>
<td>( i=n )</td>
<td>( a_{nn}^* )</td>
<td>( n )</td>
<td>( a_{n+1,n+1} = a_{1,1} )</td>
<td>( C_2 )</td>
<td>( n+1-1 = 1 )</td>
</tr>
</tbody>
</table>

In all cases the powers of \( y_1 \) and \( y_2 \) in \( a_{i+1,j+1}^* \) are the same as those in \( a_{i,j}^* \).

Hence, with the convention (36), the equality (35) is satisfied.

The result (35) is important because it means that only the \( n \) elements of \( C_3 \), the last column of \( A_1^{-1} \), need be calculated:

\[ a_{in}^* = \frac{1}{b(y_1 - y_2)} \left( \frac{y_1^i}{1-y_1^n} - \frac{y_2^i}{1-y_2^n} \right) \quad \text{for} \quad i = 1, \ldots, n \] (37)

with
\[ y_1 = \frac{-a + \sqrt{a^2 - 4bc}}{2b} \quad \text{and} \quad y_2 = \frac{-a - \sqrt{a^2 - 4bc}}{2b}, \]  

(38)

then all elements of \( A^{-1} \) are known, by making \( A^{-1} \) a circulant from that column.

Example

\[
\begin{align*}
    a &= -7 \\
    b &= 3 \\
    c &= 2 \\
    n &= 4 \\
    A &= \begin{bmatrix}
        -7 & 3 & 0 & 2 \\
        2 & -7 & 3 & 0 \\
        0 & 2 & -7 & 3 \\
        3 & 0 & 2 & -7
    \end{bmatrix}
\end{align*}
\]

\[
y_1, y_2 = \frac{7 + \sqrt{49 - 24}}{6} = \frac{7 + 5}{6} = 2, \frac{1}{3}
\]

\[
b(y_1 - y_2) = 3(2 - \frac{1}{3}) = 5
\]

\[
\frac{1}{1 - y^n_1} = \frac{1}{1 - 16} = \frac{-1}{15} \quad \frac{1}{1 - y^n_2} = \frac{1}{1 - (1/3)^4} = \frac{81}{80}
\]

\[
a^n_{1n} = \frac{1}{5} \left( \frac{-2}{15} - \frac{27}{80} \right) = \frac{-1}{1200} (32 + 81) = \frac{-113}{1200}
\]

\[
a^n_{2n} = \frac{1}{5} \left( \frac{-4}{15} - \frac{9}{80} \right) = \frac{-1}{1200} (64 + 27) = \frac{-91}{1200}
\]

\[
a^n_{3n} = \frac{1}{5} \left( \frac{-8}{15} - \frac{3}{80} \right) = \frac{-1}{1200} (128 + 9) = \frac{-137}{1200}
\]

\[
a^n_{4n} = \frac{1}{5} \left( \frac{-16}{15} - \frac{1}{80} \right) = \frac{-1}{1200} (256 + 3) = \frac{-259}{1200}
\]

These are the elements of the last column of \( A^{-1} \). Cyclic permutation of these elements gives the other columns and so
\[ A^{-1} = \frac{-1}{1200} \begin{bmatrix} 259 & 137 & 91 & 113 \\ 113 & 259 & 137 & 91 \\ 91 & 113 & 259 & 137 \\ 137 & 91 & 113 & 259 \end{bmatrix} \]

**Calculations**

Although \( y_1 - y_2 \) is a factor of the denominator of (37) it is also a factor of the numerator, as is evident from expressing this as

\[
y_1^i(1-y_2^n) - y_2^i(1-y_1^n) = (y_1^i - y_2^i) + y_1^i y_2^i (y_1^{-i} - y_2^{-i}) \]

On removing this factor from both numerator and denominator, (37) can then be used when \( y_1 = y_2 \).

The terms \( 1 - y^n \) in the denominator of (37) imply that (37) cannot be used when \( y_1 \) or \( y_2 \) are unity; i.e., excluded are cases when

\[
-a \pm \frac{\sqrt{a^2-4bc}}{2b} = 1
\]

This reduces to

\[
b(a + b + c) = 0,
\]

i.e.,

\[
a + b + c = 0 \text{ or } b = 0.
\]

It is well known that the determinant of a circulant matrix has the sum of its elements as a factor. In the case here this sum is \( a + b + c \), and when \( a + b + c = 0 \), \( |A| = 0 \) and \( A^{-1} \) does not exist. Thus the exclusion of \( a + b + c = 0 \) from (37) is consistent with the inverse not existing. Likewise, exclusion of \( b = 0 \) is consistent with \( y_1 \).
with \( y_1 \) and \( y_2 \) not existing, since they are multiples of \( 1/b \). This is because \( b = 0 \) changes the recurrence relationship (9) and leads to a different solution altogether. This is discussed subsequently, as Case 3.

Not only does \( y_1 = 1 \) make (37) meaningless but so also does \( y_1 = -1 \) if \( n \) is even. We then exclude

\[
\frac{-a \pm \sqrt{a^2 - 4bc}}{2b} = -1,
\]

which is

\[ b(a-b-c) = 0 \]

i.e.

\[ a = b + c \text{ or } b = 0. \]

The exclusion of \( a = b + c \) for \( n \) even might be considered an unexpected result. However, when \( a = b + c \), for \( n = 4 \) it is easily seen that \( A \) has no inverse. By adding all rows of

\[
|A| = \begin{bmatrix}
  a & b & 0 & c \\
  c & a & b & 0 \\
  0 & c & a & b \\
  b & 0 & c & a \\
\end{bmatrix}
\]

to the first, then subtracting the first column from all others we get

\[
|A| = (a+b+c) \begin{bmatrix}
  1 & 0 & 0 & 0 \\
  c & a-c & b-c & -c \\
  0 & c & a & b \\
  b & -b & c-b & a-b \\
\end{bmatrix} = (a+b+c) \begin{bmatrix}
  a-c & b-c & -c \\
  c & a & b \\
  -b & c-b & a-b \\
\end{bmatrix}
\]

and if \( a = b + c \)
I = (a+b+c)

\[ |A| = (a+b+c) \begin{bmatrix} b & b-c & -c \\ c & a & b \\ -b & c-b & c \end{bmatrix} = 0 . \]

Presumably this result is true for all even \( n \).

**Case 2**

\[
A = \begin{bmatrix}
  a & b & 0 & 0 & b \\
  b & a & b & 0 & 0 \\
  0 & b & a & b & 0 \\
  0 & 0 & b & a & b \\
  0 & 0 & 0 & b & a
\end{bmatrix}
\]

This is simply the symmetric form of Case 1, with \( c = b \), and hence

\[
y_1, y_2 = \frac{-a \pm \sqrt{a^2 - 4b^2}}{2b} .
\]

**Case 3**

This is Case 1 with \( b = 0 \); e.g.

\[
A_1 = \begin{bmatrix}
  a & 0 & 0 & 0 & c \\
  c & a & 0 & 0 & 0 \\
  0 & c & a & 0 & 0 \\
  0 & 0 & c & a & 0 \\
  0 & 0 & 0 & c & a
\end{bmatrix}
\]

The recurrence equation (9) is now

\[
cz_{i-1} + az_i = w^i
\]
which, with \( u_i = z_i / w^i \) is

\[
c u_{i-1} + a w u_i = 1
\]

to which the solution for \( z_i \) is

\[
z_i = w^i (\alpha x^i + \beta) = \alpha y^i + \beta w^i \tag{40}
\]

where

\[
x = -c / a w = y / w \quad \text{for} \quad y = -c / a . \tag{41}
\]

The initial condition is now just the first of (26), \( z_0 = z_n \). Using \( i = 0 \) and \( i = n \) in (40) gives

\[
\alpha = \beta (\frac{y^n - 1}{1 - y^n}) = -\frac{(1 - w^n) y^i}{1 - y^n} \tag{42}
\]

Then for \( z_1 \) and \( z_2 \) from (40) substituted in (39) we get

\[
c (\alpha y + \beta w) + a (\alpha y^2 + \beta w^2) = w^2 ,
\]

i.e.

\[
\alpha y (c + ay) + \beta w (c + aw) = w^2 ,
\]

and on using (41) this reduces to

\[
\beta = \frac{w}{c + aw} .
\]

Substituting this and (42) into (40) gives

\[
z_i = \frac{w}{c + aw} \left[ \frac{w^i}{w} - \frac{(1 - w^n)y}{1 - y^n} \right] .
\]

On noting that since \( y = -c / a \),

\[
\frac{w}{c + aw} = \frac{1}{a (1 - y / w)} = \frac{1}{a} \sum_{k=0}^{\infty} \frac{y^k}{w^k} ,
\]
we then have
\[ z_i = \frac{1}{a} \left[ \sum_{k=0}^{\infty} y^k w^{i-k} - \frac{y^i}{1-y^n} \sum_{k=0}^{\infty} y^k w^{-k} + \frac{y^i}{1-y^n} \sum_{k=0}^{\infty} y^k w^{-n-k} \right]. \]

The purpose of \( z_i \) is derivation of the coefficient of \( w^j \) for positive \( j = 1, 2, \ldots, n \). Changing subscripts, \( i-k = j \) in the first term and \( n-k = j \) in the last, and ignoring all powers of \( w \) outside the range 1 through \( n \), we then write
\[ z_i = \frac{1}{a} \left[ \sum_{j=1}^{i} y^{i-j} w^j + \frac{y^i}{1-y^n} \sum_{j=1}^{n} y^{n-j} w^j \right]. \]

Hence
\[ a_{ij}^* = \text{coefficient of } w^j \text{ in } z_i \]
is, for
\[ i \leq j : \quad a_{ij}^* = \frac{1}{a} \left( y^{i-j} + \frac{y^{n+i-j}}{1-y^n} \right) = \frac{y^{i-j}}{a(1-y^n)} \] \hspace{1cm} (43)
\[ j > i : \quad a_{ij}^* = \frac{y^{n+i-j}}{a(1-y^n)} \] \hspace{1cm} (44)

Since \( A \) and \( A^{-1} \) are circulants we need only specify elements of the last column
\[ a_{in}^* = \frac{y^i}{a(1-y^n)} \quad \text{for} \quad i < n \]
\[ a_{nn}^* = \frac{1}{a(1-y^n)} \]

with
\[ y = \frac{-c}{a}. \]
Exclusions here are $a + c = 0$ and $a = c$ for $n$ even.

**Example**

$\begin{align*}
    c &= -6 \\
    a &= 2 \\
    n &= 4 \\
\end{align*}$

$A = \begin{bmatrix}
    2 & 0 & 0 & -6 \\
    -6 & 2 & 0 & 0 \\
    0 & -6 & 2 & 0 \\
    0 & 0 & -6 & 2
\end{bmatrix}$

$y = -(-6)/2 = 3; \quad a(1-y^n) = 2(1-3^4) = -160.$

Elements $a^k_{ij}$ are either $3^{i-j}/(-160)$ or $3^{4+i-j}/(-160)$ so that

$A^{-1} = \frac{-1}{160} \begin{bmatrix}
    1 & 27 & 9 & 3 \\
    3 & 1 & 27 & 9 \\
    9 & 3 & 1 & 27 \\
    27 & 9 & 3 & 1
\end{bmatrix}$

**Case 4**

$A = \begin{bmatrix}
    a & b & t & t & t \\
    c & a & b & t & t \\
    t & c & a & b & t \\
    t & t & c & a & b \\
    b & t & t & c & a
\end{bmatrix}$

The recurrence equation here is

$$cz_{i-1} + az_i + bz_{i+1} + \sum_{k \neq i-1, i, i+1} tz_k = w^i$$
\[(c-t)z_{i-1} + (a-t)z_i + (b-t)z_{i+1} + t \sum_{i=1}^{n} z_i = w^i \quad (47)\]

The initial conditions are, as in (26)

\[z_0 = z_n \quad \text{and} \quad z_{n+1} = z_1 \quad (48)\]

Summing (47) over \(i = 1, \ldots, n\), and using (48) gives

\[(c-t + a-t + b-t) \sum_{i=1}^{n} z_i + nt \sum_{i=1}^{n} z_i = \sum_{i=1}^{n} w^i \quad (49)\]

Hence

\[\sum_{i=1}^{n} z_i = \frac{\sum_{i=1}^{n} w^i}{nt+a+b+c-3t} = \theta, \quad \text{say} \quad (49)\]

and so (47) is

\[(c-t)z_{i-1} + (a-t)z_i + (b-t)z_{i+1} + t\theta = w^i \quad (50)\]

Now define

\[\lambda = \frac{t\theta}{a+b+c-3t} \quad (51)\]

Then (50) is

\[(c-t)(z_{i-1} + \lambda) + (a-t)(z_i + \lambda) + (b-t)(z_{i+1} + \lambda) = w^i \quad (52)\]

with, from (48)

\[z_0 + \lambda = z_n + \lambda \quad \text{and} \quad z_{n+1} + \lambda = z_1 + \lambda \quad (53)\]

These are exactly the same equations as (9) and (26) except that \(z_i + \lambda\) is used in place of \(z_i\) and \(a-t, b-t\) and \(c-t\) are used in place of \(a, b,\) and \(c\). Therefore, with these replacements, the solution for \(z_i + \lambda\) to (52) is exactly (34). Hence
\[ z_i = (34) - \lambda = (34) - \frac{t \sum w^i}{(a+b+c-3t)(nt+a+b+c-3t)} \quad (54) \]

Therefore the coefficient of \( w^j \) in \( z_i \) of (54) is the same as that of (34) only using \( a-t, b-t, c-t \) in place of \( a, b, c \), and with \( t/(a+b+c-3t)(nt+a+b+c-3t) \) subtracted. Thus from (37) we have the elements of \( A^{-1} \) for \( A \) of (46) as

\[ a_{in}^f = \frac{1}{(b-t)(y_1-y_2)} \left( \frac{y_1^i}{1-y_2^n} - \frac{y_2^i}{1-y_2^n} \right) - m \]

for

\[ m = \frac{t}{(a+b+c-3t)(nt+a+b+c-3t)} \]

and

\[ y_1, y_2 = \frac{-(a-t) \pm \sqrt{(a-t)^2 - 4(b-t)(c-t)}}{2(b-t)} \]

**Example**

\[
\begin{align*}
a &= -5 \\
b &= 5 \\
c &= 4 \\
t &= 2 \\
n &= 4 \\

A &= \begin{bmatrix}
-5 & 5 & 2 & 4 \\
4 & -5 & 5 & 2 \\
2 & 4 & -5 & 5 \\
5 & 2 & 4 & -5
\end{bmatrix}
\end{align*}
\]

\[ y_1, y_2 = \frac{-(5-2) \pm \sqrt{(5-2)^2 - 4(5-2)(4-2)}}{2(5-2)} = \frac{7 \pm \sqrt{25}}{6} = 2, \frac{1}{3} \]

\[ (b-t)(y_1-y_2) = 3(2-\frac{1}{3}) = 5 \]

\[ m = \frac{t}{(-5+5+4-3(2))(5+5+4-3(2)+4(2))} = \frac{2}{-2(6)} = \frac{2}{-12} = -\frac{1}{6} \]

Since \( y_1, y_2 \) and \( (b-t)(y_1-y_2) \) are the same values as \( y_1, y_2 \) and \( b(y_1-y_2) \) of the example in Case 1, the elements of \( A^{-1} \) are the same except for subtracting \(-1/6\) from each of them. Hence
\[ a_{1n} = \frac{-113}{1200} + \frac{1}{6} = \frac{87}{1200} \]
\[ a_{2n} = \frac{-91}{1200} + \frac{1}{6} = \frac{109}{1200} \]
\[ a_{3n} = \frac{-137}{1200} + \frac{1}{6} = \frac{63}{1200} \]
\[ a_{4n} = \frac{-252}{1200} + \frac{1}{6} = \frac{-59}{1200} \]

Hence

\[ A^{-1} = \frac{1}{1200} \begin{bmatrix} -59 & 63 & 109 & 87 \\ 87 & -59 & 63 & 109 \\ 109 & 87 & -59 & 63 \\ 63 & 109 & 87 & -59 \end{bmatrix} \]

Multiplication shows that \( AA^{-1} = I \).

References

