DISTRIBUTION OF THE RECAPTURE VECTOR IN A TAG-RECAPTURE MODEL ALLOWING MORTALITY

BY

K. H. Pollock

Cornell University

BU-475-M August, 1973

## Abstract

The distribution of the recapture vector for a tag-recapture model allowing mortality (but not recruitment) is derived directly. The result (which has several possible proofs) is used in "Tests for Mortality and Recruitment in a K-sample Tag-Recapture Experiment" by K. H. Pollock, D. L. Solomon and D. S. Robson, which will appear in Biometrics, Vol. 29, No. 1 (March, 1974).

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## SUMMARY

The distribution of the recapture vector for a tag-recapture model allowing mortality (but not recruitment) is derived directly. The result (which has several possible proofs) is used in "Tests for Mortality and Recruitment in a K-sample Tag-Recapture Experiment" by K. H. Pollock, D. L. Solomon and D. S. Robson, which will appear in Biometrics, Vol. 29, No. 1 (March, 1974).

## INTRODUCTION

In a K-sample tag-recapture experiment, a population is randomly sampled K successive times. The untagged elements of a sample are serially tagged, and the serial number of each of the previously tagged elements is recorded. The entire sample is then returned to the population. It is assumed that the force of mortality is the same for tagged and untagged elements. The sample sizes at each time will be treated as fixed, observable numbers. The population size at each time is regarded as a fixed but unknown parameter. Now we need some definitions:

$N_i$ = the population size at the $i^{th}$ sample time, $i=1,2,\cdots,K-1$ [1]

$\underset{\sim}{N} = (N_1,N_2,\cdots,N_{K-1})$

$n_i$ = the number of elements in the $i^{th}$ sample

$n_\cdot = \sum_{i=1}^{K} n_i$

[1] The parameter $N_K$ is not identifiable.

$X_{i1}$ = the number of elements captured in a sample having capture history i

e.g. $X_{101}$ = the number of elements of the $n_3$ captured in the third sample which were captured in sample one but not sample two

$N_j^{(i)}$ = the number of elements with a particular recapture history that survive until the next period where i is the recapture history and j is the next period

e.g. $N_4^{(101)}$ = the number of $X_{101}$ elements which survive until the time of the fourth sampling period

$R_i$ = the number of elements in the $i^{th}$ sample that are later recaptured at least once, $i=1,2,\cdots,K-1$ ($R_K \equiv 0$)

$\underset{\sim}{R}$ = $(R_1,R_2,\cdots,R_{K-1})$ - the recapture vector

$C_i$ = $\sum_{j=i+1}^{K} (n_j-R_j)$ = the number of distinct elements in all samples after the $i^{th}$, $i=0,1,\cdots,K-1$, where $n_j-R_j$ is the number of elements caught in the $j^{th}$ sample and subsequently not recaptured

$S_i$ = the number of marked animals in the $i^{th}$ sample

In this paper the distribution of $\underset{\sim}{R}$ is derived for K=4. In this case we have:

$$R_1 = X_{11} + X_{101} + X_{1001}$$

$$R_2 = X_{111} + X_{011} + X_{1101} + X_{0101}$$

$$R_3 = X_{1111} + X_{1011} + X_{0111} + X_{0011}$$

and

$$C_0 = n_1 + n_2 + n_3 + n_4 - R_1 - R_2 - R_3$$

$$C_1 = n_2 + n_3 + n_4 - R_2 - R_3$$

$$C_2 = n_3 + n_4 - R_3$$

$$C_3 = n_4$$

## DISTRIBUTION OF THE RECAPTURE VECTOR ($\underset{\sim}{R}$)

First a series of conditional distributions are derived which involve the components of the $R_i$ (i=1,2,3). The distributions are all of hypergeometric form.

$$P\left(N_2^{(1)}\right) = \frac{\binom{n_1}{N_2^{(1)}}\binom{N_1-n_1}{N_2-N_2^{(1)}}}{\binom{N_1}{N_2}}$$

$$P\left(X_{11}\,|\,N_2^{(1)}\right) = \frac{\binom{N_2^{(1)}}{X_{11}}\binom{N_2-N_2^{(1)}}{n_2-X_{11}}}{\binom{N_2}{n_2}}$$

$$P\left(N_3^{(11)},N_3^{(10)},N_3^{(01)}\,|\,X_{11},N_2^{(1)}\right) = \frac{\binom{X_{11}}{N_3^{(11)}}\binom{N_2^{(1)}-X_{11}}{N_3^{(10)}}\binom{n_2-X_{11}}{N_3^{(01)}}\binom{N_2-N_2^{(1)}-n_2+X_{11}}{N_3-N_3^{(1)}}}{\binom{N_2}{N_3}}$$

$$P\left(X_{111},X_{101},X_{011}\,|\,N_3^{(11)},N_3^{(10)},N_3^{(01)},X_{11},N_2^{(1)}\right) = \frac{\binom{N_3^{(11)}}{X_{111}}\binom{N_3^{(10)}}{X_{101}}\binom{N_3^{(01)}}{X_{011}}\binom{N_3-N_3^{(1)}}{n_3-S_3}}{\binom{N_3}{n_3}}$$

where

$$N_3^{(1)} = N_3^{(11)} + N_3^{(10)} + N_3^{(01)}$$

$$P\left(N_4^{(111)},N_4^{(110)},N_4^{(101)},N_4^{(100)},N_4^{(011)},N_4^{(010)},N_4^{(001)}\,|\,X_{111},\cdots,N_3^{(11)},\cdots,X_{11},N_2^{(1)}\right)$$

$$= \frac{\binom{X_{111}}{N_4^{(111)}}\binom{N_3^{(11)}-X_{111}}{N_4^{(110)}}\binom{X_{101}}{N_4^{(101)}}\binom{N_3^{(10)}-X_{101}}{N_4^{(100)}}\binom{X_{011}}{N_4^{(011)}}\binom{N_3^{(01)}-X_{011}}{N_4^{(010)}}\binom{n_3-S_3}{N_4^{(001)}}\binom{N_3-N_3^{(1)}-n_3+S_3}{N_4-N_4^{(1)}}}{\binom{N_3}{N_4}}$$

$$P\left(X_{1111},X_{1101},X_{1011},X_{1001},X_{0111},X_{0101},X_{0011}\,|\,N_4^{(111)},\cdots,X_{111},\cdots,N_3^{(11)},\cdots,X_{11},N_2^{(1)}\right)$$

$$= \frac{\binom{N_4^{(111)}}{X_{1111}}\binom{N_4^{(110)}}{X_{1101}}\binom{N_4^{(101)}}{X_{1011}}\binom{N_4^{(100)}}{X_{1001}}\binom{N_4^{(011)}}{X_{0111}}\binom{N_4^{(010)}}{X_{0101}}\binom{N_4^{(001)}}{X_{0011}}\binom{N_4-N_4^{(1)}}{n_4-S_4}}{\binom{N_4}{n_4}}$$

Now to get the distribution of $\underset{\sim}{R}$ we have to sum out the $N_i^{(j)}$'s (conditional on $\sum_j N_i^{(j)} = N_i$) and then sum over the components of each $R_i$. To ease computation we shall do both steps together in stages.

$$P\left(X_{1111},X_{1101},\cdots,X_{0011}\,|\,X_{111},\cdots,N_3^{(11)},\cdots,X_{11},N_2^{(1)}\right)$$

$$= \left[\frac{\binom{X_{111}}{X_{1111}}\binom{N_3^{(11)}-X_{111}}{X_{1101}}\binom{X_{101}}{X_{1011}}\binom{N_3^{(10)}-X_{101}}{X_{1001}}\binom{X_{011}}{X_{0111}}\binom{N_3^{(01)}-X_{011}}{X_{0101}}\binom{n_3-S_3}{X_{0011}}\binom{N_3-N_3^{(1)}-n_3+S_3}{n_4-S_4}}{\binom{N_3}{N_4}\binom{N_4}{n_4}}\right]$$

$$\cdot \sum_{N_4^{(111)},\cdots,N_4^{(001)}} \binom{X_{111}-X_{1111}}{N_4^{(111)}-X_{1111}}\binom{N_3^{(11)}-X_{111}-X_{1101}}{N_4^{(110)}-X_{1101}}\cdots\binom{n_3-S_3-X_{0011}}{N_4^{(001)}-X_{0011}}\binom{N_3-N_3^{(1)}-n_3+S_3-n_4+S_4}{N_4-N_4^{(1)}-n_4+S_4}$$

$$= \left[\quad\right]\cdot\binom{N_3-n_4}{N_4-n_4}$$

$$= \frac{\binom{X_{111}}{X_{1111}}\binom{N_3^{(11)}-X_{111}}{X_{1101}}\binom{X_{101}}{X_{1011}}\binom{N_3^{(10)}-X_{101}}{X_{1001}}\binom{X_{011}}{X_{0111}}\binom{N_3^{(01)}-X_{011}}{X_{0101}}\binom{n_3-S_3}{X_{0011}}\binom{N_3-N_3^{(1)}-n_3+S_3}{n_4-S_4}}{\binom{N_3}{C_3}}$$

Now $R_3 = X_{1111} + X_{1011} + X_{0111} + X_{0011}$ and let $R_2^{(1)} = X_{1101} + X_{0101}$ and $R_2^{(0)} = X_{111} + X_{011}$. Thus

$$P\left(R_3, R_2^{(1)}, X_{1001} \mid X_{111}, \cdots, N_3^{(11)}, \cdots, X_{11}, N_2^{(1)}\right)$$

$$= \frac{\binom{n_3}{R_3}\binom{N_3^{(11)}+N_3^{(01)}-R_2^{(0)}}{R_2^{(1)}}\binom{N_3^{(10)}-X_{101}}{X_{1001}}\binom{N_3-N_3^{(1)}-n_3+S_3}{n_4-R_3-R_2^{(1)}-X_{1001}}}{\binom{N_3}{C_3}}$$

$$P\left(R_3, R_2^{(1)}, R_2^{(0)}, X_{1001}, X_{101} \mid X_{11}, N_2^{(1)}\right)$$

$$= \left[\frac{\binom{n_3}{R_3}\binom{N_2^{(1)}-X_{11}}{X_{101}, X_{1001}}\binom{n_2}{R_2^{(0)}, R_2^{(1)}}\binom{N_2-N_2^{(1)}-n_2+X_{11}}{n_3-X_{101}-R_2^{(0)}, n_4-R_3-R_2^{(1)}-X_{1001}}}{\binom{N_3}{C_3}\binom{N_2}{N_3}\binom{N_3}{n_3}}\right]$$

$$\cdot \sum_{N_3^{(11)}, N_3^{(10)}, N_3^{(01)}} \binom{n_2-R_2}{N_3^{(11)}+N_3^{(01)}-R_2}\binom{N_2^{(1)}-X_{11}-X_{101}-X_{1001}}{N_3^{(10)}-X_{101}-X_{1001}}$$

$$\cdot \binom{N_2-N_2^{(1)}-n_2+X_{11}-n_3-n_4+R_2+R_3+X_{101}+X_{1001}}{N_3-N_3^{(1)}-n_3-n_4+R_2+R_3+X_{101}+X_{1001}}$$

$$= \left[\quad\right] \cdot \binom{N_2-C_2}{N_3-C_2} \quad \text{where} \quad C_2 = n_3 + n_4 - R_3$$

$$= \frac{\binom{n_3}{R_3}\binom{N_2^{(1)}-X_{11}}{X_{101}, X_{1001}}\binom{n_2}{R_2^{(0)}, R_2^{(1)}}\binom{N_2-N_2^{(1)}-n_2+X_{11}}{n_3-R_2^{(0)}-X_{101}, n_4-R_3-R_2^{(1)}-X_{1001}}\binom{N_3}{C_2}}{\binom{N_3}{C_3}\binom{N_2}{C_2}\binom{N_3}{n_3}}$$

$$= \left[ \frac{\binom{n_3}{R_3}\binom{n_2}{R_2}\binom{N_2^{(1)}-X_{11}}{X_{101},X_{1001}}\binom{N_3}{C_2}}{\binom{N_3}{C_3}\binom{N_2}{C_2}\binom{N_3}{n_3}} \cdot \frac{R_2!\,(N_2-N_2^{(1)}-n_2+X_{11})!}{(N_2-N_2^{(1)}-C_1+R_1)!\,(n_3-X_{101})!\,(n_3-R_3-X_{1001})!} \right]$$

$$\cdot \binom{n_3-X_{101}}{R_2^{(0)}}\binom{n_4-R_3-X_{1001}}{R_2^{(1)}}$$

Thus we have

$$P\left(R_3,R_2,X_{101},X_{1001}\,|\,X_{11},N_2^{(1)}\right)$$

$$= \left[\quad\right] \cdot \binom{C_2-X_{101}-X_{1001}}{R_2}$$

$$= \left[ \frac{\binom{n_3}{R_3}\binom{n_2}{R_2}\binom{N_3}{C_2}\binom{C_2-X_{101}-X_{1001}}{R_2}}{\binom{N_3}{C_3}\binom{N_2}{C_2}\binom{N_3}{n_3}} \cdot \frac{R_2!\,(N_2-N_2^{(1)}-n_2+X_{11})!\,(N_2^{(1)}-X_{11})!}{(N_2-N_2^{(1)}-C_1+R_1)!\,(N_2^{(1)}-R_1)!\,n_3!\,(n_4-R_3)!} \right] \cdot \binom{n_3}{X_{101}}\binom{n_4-R_3}{X_{1001}}$$

Let $R_1^{(1)} = X_{101} + X_{1001}$

$$P\left(R_3,R_2,R_1^{(1)}\,|\,X_{11},N_2^{(1)}\right)$$

$$= \left[\quad\right] \cdot \binom{C_2}{R_1^{(1)}}$$

$$= \frac{\binom{n_3}{R_3}\binom{n_2}{R_2}\binom{N_3}{C_2}}{\binom{N_3}{C_3}\binom{N_2}{C_2}\binom{N_3}{n_3}} \cdot \frac{C_2!\,(N_2^{(1)}-X_{11})!\,(N_2-N_2^{(1)}-n_2+X_{11})!}{(C_2-R_2-R_1^{(1)})!\,R_1^{(1)}!\,n_3!\,(n_4-R_3)!\,(N_2^{(1)}-R_1)!\,(N_2-N_2^{(1)}-C_1+R_1)!}$$

$P\left(R_3, R_2, R_1^{(1)}, X_{11}\right)$

$$= \left[ \frac{\binom{n_3}{R_3}\binom{n_2}{R_2}\binom{N_3}{C_2}\binom{n_1}{R_1}\binom{N_1-n_1}{C_1-R_1}\binom{C_2}{n_3}\binom{C_2-R_2}{R_1^{(1)}}}{\binom{N_3}{C_3}\binom{N_2}{C_2}\binom{N_3}{n_3}\binom{N_1}{N_2}\binom{N_2}{n_2}} \cdot \frac{R_1!(C_1-R_1)!}{X_{11}!(n_2-X_{11})!(C_2-R_2)!} \right]$$

$$\cdot \sum_{N_2^{(1)}} \binom{n_1-R_1}{N_2^{(1)}-R_1}\binom{N_1-C_0}{N_2-N_2^{(1)}-C_1+R_1}$$

$$= \left[ \qquad \right] \binom{N_1-C_1}{N_2-C_1}$$

$$= \left[ \frac{\binom{n_3}{R_3}\binom{n_2}{R_2}\binom{n_1}{R_1}\binom{N_1-n_1}{C_1-R_1}\binom{N_3}{C_2}\binom{N_2}{C_1}\binom{C_2}{n_3}}{\binom{N_3}{C_3}\binom{N_2}{C_2}\binom{N_1}{C_1}\binom{N_3}{n_3}\binom{N_2}{n_2}} \cdot \frac{R_1!(C_1-R_1)!}{n_2!(C_2-R_2)!} \right] \cdot \binom{n_2}{X_{11}}\binom{C_2-R_2}{R_1^{(1)}}$$

Now $R_1 = X_{11} + R_1^{(1)}$ . Thus

$P(R_3, R_2, R_1)$

$$= \left[ \qquad \right] \cdot \binom{C_1}{R_1}$$

$$= \left[ \frac{\binom{n_3}{R_3}\binom{n_2}{R_2}\binom{n_1}{R_1}\binom{N_1-n_1}{C_1-R_1}}{\binom{N_3}{C_3}\binom{N_2}{C_2}\binom{N_1}{C_1}} \right] \cdot \frac{\binom{N_3}{C_2}\binom{N_2}{C_1}\binom{C_2}{n_3}}{\binom{N_3}{n_3}\binom{N_2}{n_2}} \cdot \frac{C_1!}{n_2!(C_2-R_2)!}$$

If we expand all the terms outside the square brackets and cancel out terms we obtain

$$P(R_3,R_2,R_1)$$

$$= \Big[ \qquad \Big] \cdot \frac{(N_3-n_3)!}{(N_3-C_2)!(C_3-R_3)} \cdot \frac{(N_2-n_2)!}{(N_2-C_1)!(C_2-R_2)!}$$

Using relations for $C_2$ and $C_3$ given earlier

$$P(R_3,R_2,R_1) = \Big[ \qquad \Big] \cdot \binom{N_3-n_3}{C_3-R_3}\binom{N_2-n_2}{C_2-R_2}$$

Finally we obtain the desired result

$$P(\underset{\sim}{R};\underset{\sim}{N}) = \prod_{i=1}^{3} \frac{\binom{n_i}{R_i}\binom{N_i-n_i}{C_i-R_i}}{\binom{N_i}{C_i}}$$