ON CONNECTEDNESS IN TWO-WAY ELIMINATION
OF HETEROGENEITY DESIGNS

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ABSTRACT

A necessary and sufficient condition is established for doubly-connectedness in \( b \)-row and \( k \)-column designs in which all cells are filled. An algorithm is presented for constructing a class of doubly-disconnected designs which are pairwise connected with respect to rows, columns, and treatments. A necessary condition for doubly-connectedness in a generalized two-way elimination of heterogeneity designs is provided and a property of doubly-connected designs is given.

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1. Introduction. Let the experimental material be arranged in \( b \) rows and \( k \) columns and let \( v \) treatments be applied to the experimental units, the \( i^{th} \) treatment being replicated \( r_i \) times \((i = 1, 2, \ldots, v)\). Let \( u_1, u_2, \ldots, u_b \) units be treated with the treatments in the \( b \) rows and let \( w_1, w_2, \ldots, w_k \) units be treated with the treatments in the \( k \) columns. If \( u_1 = u_2 = \ldots = u_b = k \) and \( w_1 = w_2 = \ldots = w_k = b \) we obtain the usual two-way elimination of heterogeneity designs, otherwise we obtain generalized two-way elimination of heterogeneity designs.

Let \( N = (n_{ij}) \) be a \( b \times k \) matrix, where \( n_{ij} \) is the number of treated units in the \( i^{th} \) row and \( j^{th} \) column, \( L = (l_{ij}) \) be a \( v \times b \) matrix, where \( l_{ij} \) is the number of treated units with the \( i^{th} \) treatment in the \( j^{th} \) row and \( M = (m_{ij}) \) be a \( v \times k \) matrix, where \( m_{ij} \) is the number of treated units with the \( i^{th} \) treatment in the \( j^{th} \) column. Let \( A^{-} \) denote a generalized inverse of \( A \). Let \( C_1, C_2, C_3, C_4, C^*, C_3^*, C_4^* \) be the well-known \( C \) matrices for the following purposes:

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Key words: doubly-connected design, two-way design, pairwise connected design.
<table>
<thead>
<tr>
<th>Matrix</th>
<th>Estimating</th>
<th>Eliminating</th>
<th>Ignoring</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_1 = \text{diag}(r_1, r_2, \ldots, r_v)$ - $L \text{diag}(\frac{1}{u_1}, \frac{1}{u_2}, \ldots, \frac{1}{u_b}) L'$</td>
<td>treatment effects effects effects</td>
<td>row column effects effects effects</td>
<td></td>
</tr>
<tr>
<td>$C_2 = \text{diag}(r_1, r_2, \ldots, r_v)$ - $M \text{diag}(\frac{1}{w_1}, \frac{1}{w_2}, \ldots, \frac{1}{w_k}) M'$</td>
<td>treatment column row effects effects effects</td>
<td>column row treatment effects effects effects</td>
<td></td>
</tr>
<tr>
<td>$C_3 = \text{diag}(w_1, w_2, \ldots, w_k)$ - $N' \text{diag}(\frac{1}{u_1}, \frac{1}{u_2}, \ldots, \frac{1}{u_b}) N$</td>
<td>row column row effects effects effects</td>
<td>column row treatment effects effects effects</td>
<td></td>
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<tr>
<td>$C_4 = \text{diag}(u_1, u_2, \ldots, u_b)$ - $N \text{diag}(\frac{1}{w_1}, \frac{1}{w_2}, \ldots, \frac{1}{w_k}) N'$</td>
<td>row column row effects effects effects</td>
<td>column row treatment effects effects effects</td>
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<tr>
<td>$C^* = C_1 - (M-L \text{diag}(\frac{1}{u_1}, \frac{1}{u_2}, \ldots, \frac{1}{u_b}) N) x C_3 (M-L \text{diag}(\frac{1}{u_1}, \frac{1}{u_2}, \ldots, \frac{1}{u_b}) N)$</td>
<td>treatment row and column effects column effects - effects</td>
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<tr>
<td>$C^*_3 = C_3 - (M-L \text{diag}(\frac{1}{u_1}, \frac{1}{u_2}, \ldots, \frac{1}{u_b}) N) x C_1 (M-L \text{diag}(\frac{1}{u_1}, \frac{1}{u_2}, \ldots, \frac{1}{u_b}) N)$</td>
<td>column row and column effects treatment effects - effects</td>
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<tr>
<td>$C^*_4 = C_4 - (L-M \text{diag}(\frac{1}{w_1}, \frac{1}{w_2}, \ldots, \frac{1}{w_k}) N') x C_1 (L-M \text{diag}(\frac{1}{w_1}, \frac{1}{w_2}, \ldots, \frac{1}{w_k}) N')$</td>
<td>row column and column effects treatment effects - effects</td>
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</table>

$C^*$ in case of an ordinary two-way elimination of heterogeneity designs reduces to the following:

\[(1.1) \quad C^* = \text{diag}(r_1, r_2, \ldots, r_v) - \frac{1}{k} LL' - \frac{1}{b} MM' + \frac{1}{bk} rr',\]

where $r' = (r_1, r_2, \ldots, r_v)$. 
The design is said to be row-treatment connected if $R(C_1) = v-1$, column-treatment connected if $R(C_2) = v-1$, column-row connected if $R(C_3) = k-1$, and doubly connected if $R(C^*) = v-1$. It may be noted that a doubly connected design need not necessarily ensure the estimation of all elementary contrasts of row and column effects.

This paper is a contribution toward the theory of the doubly-connectedness property of two-way elimination of heterogeneity designs.

2. A necessary and sufficient condition for doubly-connectedness of ordinary two-way elimination of heterogeneity designs. Before we state and prove our main theorem, we state the following lemma whose proof is obvious.

**Lemma 2.1.** The $C$-matrices, $C_1, C_2, C^*$ have rank $v-1$ if and only if $C_1 + aJ_{v,v}$, $C_2 + aJ_{v,v}$, $C^* + aJ_{v,v}$ are non-singular, where $a$ is a non-zero scalar and $J_{m,n}$ is an $m \times n$ matrix with $1$ everywhere.

**Theorem 2.1.** In an ordinary two-way elimination of heterogeneity design, let $r_1 = r_2 = \cdots = r_v = r$ and let $L'M = rJ_{b,k}$. Then the design is doubly-connected if and only if it is row-treatment and column-treatment connected.

**Proof.** Under the assumptions of the theorem

$$C^* = rC_1C_2,$$

$$C^* + aJ_{v,v} = r(C_1 + \sqrt{\frac{a}{rv}} J_{v,v})C_2 + \sqrt{\frac{a}{rv}} J_{v,v}$$

for any non-zero real $a$. Thus the following two-way relations establishing the theorem hold:
The design is doubly-connected \( R(C^*) = v - 1 \) \( \Rightarrow C^* + a_{ij} \) \( \Rightarrow v_{ij} \) is non-singular.

\[ C_1 + \sqrt{a_{ij}} v_{ij} \text{ and } C_2 + \sqrt{a_{ij}} v_{ij} \text{ are non-singular} \]

\[ R(C_1) = v - 1 \text{ and } R(C_2) = v - 1 \]

The design is row-treatment and column-treatment connected.

The conditions of the theorem are certainly much stronger than what is needed, for the design

\[
\begin{array}{cccc}
1 & 1 & 2 & 3 \\
2 & 3 & 1 & 2 \\
\end{array}
\]

is row-treatment, column-treatment, row-column and doubly-connected without satisfying the assumptions of the theorem. However, some conditions are needed as the following design

\[
\begin{array}{cc}
1 & 2 \\
2 & 3 \\
\end{array}
\]

does not satisfy the assumptions or the conclusions of the theorem.

3. An algorithm for constructing series of doubly-disconnected designs, which are pairwise connected. Consider an \( s \times s \) latin square design where the diagonal elements are \( 1, 2, \ldots, s \) in that order. When \( s \neq 2 \), the construction of such designs, in general, was provided by Hedayat and Federer [1970]. By omitting the diagonal treatments excepting the first one, we obtain a doubly-disconnected design, which is pairwise connected with respect to row, column, and treatments. In fact if \( I_n \) is an identity matrix, for such a design
\[
L = M = N = L' = M' = N' = \begin{bmatrix}
1 & J_{1,s-1} \\
J_{s-1,1} & J_{s-1,s-1} - I_{s-1}
\end{bmatrix}
\]

\[u_1 = w_1 = r_1 = s\]

\[u_2 = \cdots = u_s = w_2 = \cdots = w_s = r_2 = \cdots = r_s = s-1\]

\[c_1 = c_2 = c_3 = M - L \ \text{diag}\left(\frac{1}{u_1}, \frac{1}{u_2}, \ldots, \frac{1}{u_s}\right) N\]

so that

\[c^* = c_1 - c_3 c_2 c_3 = c_1 - c_3 = 0_{s,s}\]

where \(0_{s,s}\) is the null matrix of order \(s\).

4. A necessary condition for doubly-connectedness in generalized two-way elimination of heterogeneity designs. Contrary to the belief that a doubly-connected design is pairwise connected, the following result holds:

**Theorem 4.1.** A doubly-connected generalized two-way elimination of heterogeneity design is row-treatment and column-treatment connected.

**Proof.** Let if possible a doubly-connected design be row-treatment disconnected. Then there exists a \(v \times 1\) column vector \(\xi\) orthogonal to \(J_{s-1,1}\) such that \(c_1 \xi = 0_{v,1}\). Then, we have

\[(4.1) \quad \xi' c^* \xi = - \left\{\left(\frac{1}{u_1}, \frac{1}{u_2}, \ldots, \frac{1}{u_s}\right) N\right\} \xi' c_3^{-1}
\]

\[\times \left\{\left(\frac{1}{u_1}, \frac{1}{u_2}, \ldots, \frac{1}{u_s}\right) N\right\} \xi\]

which is a contradiction as the left-hand side is a positive quantity and the
right-hand side is a non-positive quantity in view of \( C^* \) and \( C_3^- \) being positive semi-definite matrices. Thus a doubly-connected design is row-treatment connected. Analogously, one may show it to be column-treatment connected.

The above result can also be obtained from the model of such designs. A doubly-connected design need not necessarily be row-column connected. The following design where X's indicate blanks is doubly-connected and hence row-treatment and column-treatment connected, but is row-column disconnected:

\[
\begin{array}{cccc}
1 & 2 & X & X \\
2 & 1 & X & X \\
X & X & 1 & 2 \\
X & X & 2 & 1 \\
\end{array}
\]

In fact, the class of doubly-connected designs \( I \otimes \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \) is row-column disconnected, where \( I \) is the identity matrix and \( \otimes \) is the symbol for the Kronecker product of matrices.

5. A property of doubly-connected designs. One may wonder whether a doubly-connected design can be used to estimate every elementary contrast of row and column effects. The answer is provided by the following theorem:

**Theorem 5.1.** A doubly-connected design, which is also row-column connected, provides estimates for every elementary contrast of row and column effects.

**Proof.** For a doubly-connected design which is row-column connected, the following hold:

\[
\begin{align*}
R(C^*) &= R(C_1) = R(C_2) = v - 1 \\
R(C_3) &= k - 1 \\
R(C_4) &= b - 1
\end{align*}
\]  
(5.1)
and thus

\[
\begin{bmatrix}
\text{diag}(u_1, u_2, \ldots, u_b) & N & L' \\
N' & \text{diag}(w_1, w_2, \ldots, w_k) & M' \\
L & M & \text{diag}(r_1, r_2, \ldots, r_v)
\end{bmatrix}
\]

(5.2)

\[
= R \begin{bmatrix}
0 & C_3 & * \\
0 & 0 & C^*
\end{bmatrix}
\]

\[
= b + k - l + v - 1 = v + b + k - 2,
\]

where * denotes a matrix obtained in the sweeping-out process. Again

\[
\begin{bmatrix}
\text{diag}(u_1, u_2, \ldots, u_b) & N & L' \\
N' & \text{diag}(w_1, w_2, \ldots, w_k) & M' \\
L & M & \text{diag}(r_1, r_2, \ldots, r_v)
\end{bmatrix}
\]

(5.3)

\[
= R \begin{bmatrix}
C_4^* & 0 & 0 \\
N' & \text{diag}(w_1, w_2, \ldots, w_k) & M' \\
* & 0 & C_2
\end{bmatrix}
\]

\[
= R(C_4^*) + k + v - 1,
\]

from which it follows that

\[
R(C_4^*) = b - 1,
\]

(5.4)

and
\[ v+b+k-2 = R \begin{bmatrix} \text{diag}(u_1, u_2, \ldots, u_b) & N & L' \\ N' & \text{diag}(w_1, w_2, \ldots, w_k) & M' \\ L & M & \text{diag}(r_1, r_2, \ldots, r_v) \end{bmatrix} \]

(5.5)

\[ \begin{bmatrix} \text{diag}(u_1, u_2, \ldots, u_b) & N & L' \\ 0 & C_3^* & 0 \\ 0 & 0 & C_1 \end{bmatrix} = R \]

\[ = R(C_3^*) + b + v - l , \]

from which it follows that

(5.6)

\[ R(C_3^*) = k - l . \]

Thus all elementary contrasts of row and column effects are estimable establishing the theorem.

6. **Concluding remarks**: Though the results in this paper are obtained for generalized two-way elimination of heterogeneity designs, they can be translated into the terminology of any 3-factor experiment without any loss of generality in an obvious way.

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**REFERENCE**