HYPOTHESIS TESTING IN RESTRICTED LINEAR MODELS: CORRECTING AN ERROR

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BU-451-M

March, 1973

Revised October, 1978

Abstract

The numerator sum of squares in the F-statistic for testing a hypothesis in a linear model containing restrictions involving estimable functions is incorrect in Searle [1971]. The correction has been provided by Timm and Carlson [1973], for the full rank model. Details for the non-full rank model are given here.

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The restricted linear model we consider is $\underset{\sim}{y} = \underset{\sim}{X}\underset{\sim}{b} + \underset{\sim}{e}$ subject to restrictions $\underset{\sim}{P}'\underset{\sim}{b} = \underset{\sim}{\delta}$ where $\underset{\sim}{P}'\underset{\sim}{b}$ is a set of r linearly independent estimable functions. For this model Searle [1971, p. 206] shows that a solution vector for $\underset{\sim}{b}$ is

$$\underset{\sim}{b}_r^o = \underset{\sim}{b}^o - \underset{\sim}{G}\underset{\sim}{P}(\underset{\sim}{P}'\underset{\sim}{G}\underset{\sim}{P})^{-1}(\underset{\sim}{P}'\underset{\sim}{b}^o - \underset{\sim}{\delta}) \tag{1}$$

where $\underset{\sim}{b}^o = \underset{\sim}{G}\underset{\sim}{X}'\underset{\sim}{y}$ and $\underset{\sim}{X}'\underset{\sim}{X}\underset{\sim}{G}\underset{\sim}{X}'\underset{\sim}{X} = \underset{\sim}{X}'\underset{\sim}{X}$. The error sum of squares given on the same page is

$$SSE_r = SSE + (\underset{\sim}{P}'\underset{\sim}{b}^o - \underset{\sim}{\delta})'(\underset{\sim}{P}'\underset{\sim}{G}\underset{\sim}{P})^{-1}(\underset{\sim}{P}'\underset{\sim}{b}^o - \underset{\sim}{\delta}) \tag{2}$$

where $SSE = \underset{\sim}{y}'[\underset{\sim}{I} - \underset{\sim}{X}(\underset{\sim}{X}'\underset{\sim}{X})^-\underset{\sim}{X}']\underset{\sim}{y}$.

In testing the hypothesis $H: \underset{\sim}{K}'\underset{\sim}{b} = \underset{\sim}{m}$ where $\underset{\sim}{K}'\underset{\sim}{b}$ represents s linearly independent estimable functions, we confine attention to the case where the row spaces of $\underset{\sim}{P}'$ and $\underset{\sim}{K}'$ are linearly independent. On defining

$$\mathbf{Q}' = \begin{bmatrix}\mathbf{P}'\\ \mathbf{K}'\end{bmatrix} \quad \text{and} \quad \boldsymbol{\ell} = \begin{bmatrix}\boldsymbol{\delta}\\ \mathbf{m}\end{bmatrix}, \tag{3}$$

p. 207 of Searle [1971] then gives $\mathbf{b}^{\circ}_{r,H} = \mathbf{b}^{\circ} - \mathbf{GQ}(\mathbf{Q}'\mathbf{GQ})^{-1}(\mathbf{Q}'\mathbf{b}^{\circ} - \boldsymbol{\ell})$ and

$$SSE_{r,H} = SSE + (\mathbf{Q}'\mathbf{b}^{\circ} - \boldsymbol{\ell})'(\mathbf{Q}'\mathbf{GQ})^{-1}(\mathbf{Q}'\mathbf{b}^{\circ} - \boldsymbol{\ell}) . \tag{4}$$

The F-statistic for testing H is then derived in Searle [1971] with numerator

$$SSE_{r,H} - SSE = (\mathbf{Q}'\mathbf{b}^{\circ} - \boldsymbol{\ell})'(\mathbf{Q}'\mathbf{GQ})^{-1}(\mathbf{Q}'\mathbf{b}^{\circ} - \boldsymbol{\ell}) .$$

This, as Timm and Carlson [1973, p. 33] point out, is wrong. It should be

$$SSE_{r,H} - SSE_r = (\mathbf{Q}'\mathbf{b}^{\circ} - \boldsymbol{\ell})'(\mathbf{Q}'\mathbf{GQ})^{-1}(\mathbf{Q}'\mathbf{b}^{\circ} - \boldsymbol{\ell}) - (\mathbf{P}'\mathbf{b}^{\circ} - \boldsymbol{\delta})'(\mathbf{P}'\mathbf{GP})^{-1}(\mathbf{P}'\mathbf{b}^{\circ} - \boldsymbol{\delta}) . \tag{5}$$

We show how this simplifies. From (3)

$$(\mathbf{Q}'\mathbf{GQ})^{-1} = \begin{bmatrix}\mathbf{P}'\mathbf{GP} & \mathbf{P}'\mathbf{GK}\\ \mathbf{K}'\mathbf{GP} & \mathbf{K}'\mathbf{GK}\end{bmatrix}^{-1} = \begin{bmatrix}(\mathbf{P}'\mathbf{GP})^{-1} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix} + \begin{bmatrix}-(\mathbf{P}'\mathbf{GP})^{-1}\mathbf{P}'\mathbf{GK}\\ \mathbf{I}\end{bmatrix}\mathbf{W}^{-1}[-\mathbf{K}'\mathbf{GP}(\mathbf{P}'\mathbf{GP})^{-1} \quad \mathbf{I}]$$

with $\mathbf{W} = \mathbf{K}'\mathbf{GK} - \mathbf{K}'\mathbf{GP}(\mathbf{P}'\mathbf{GP})^{-1}\mathbf{P}'\mathbf{GK}$. Hence, using (3) again, (5) becomes

$$\begin{aligned} SSE_{r,H} - SSE_r &= [(\mathbf{P}'\mathbf{b}^{\circ} - \boldsymbol{\delta})' \quad (\mathbf{K}'\mathbf{b}^{\circ} - \mathbf{m})'] \\ &\quad \times \left\{ \begin{bmatrix}(\mathbf{P}'\mathbf{GP})^{-1} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix} + \begin{bmatrix}-(\mathbf{P}'\mathbf{GP})^{-1}\mathbf{P}'\mathbf{GK}\\ \mathbf{I}\end{bmatrix}\mathbf{W}^{-1}[-\mathbf{K}'\mathbf{GP}(\mathbf{P}'\mathbf{GP})^{-1} \quad \mathbf{I}] \right\} \begin{bmatrix}\mathbf{P}'\mathbf{b}^{\circ} - \boldsymbol{\delta}\\ \mathbf{K}'\mathbf{b}^{\circ} - \mathbf{m}\end{bmatrix} \\ &\quad - (\mathbf{P}'\mathbf{b}^{\circ} - \boldsymbol{\delta})(\mathbf{P}'\mathbf{GP})^{-1}(\mathbf{P}'\mathbf{b}^{\circ} - \boldsymbol{\delta}) \\ &= \boldsymbol{\tau}'\mathbf{W}^{-1}\boldsymbol{\tau} \quad \text{with } \boldsymbol{\tau} = \mathbf{K}'\mathbf{b}^{\circ} - \mathbf{m} - \mathbf{K}'\mathbf{GP}(\mathbf{P}'\mathbf{GP})^{-1}(\mathbf{P}'\mathbf{b}^{\circ} - \boldsymbol{\delta}) \\ &\qquad\qquad = \mathbf{K}'\mathbf{b}^{\circ} - \mathbf{m} + \mathbf{K}'(\mathbf{b}^{\circ}_r - \mathbf{b}^{\circ}) \quad \text{from (2)} \\ &\qquad\qquad = (\mathbf{K}'\mathbf{b}^{\circ}_r - \mathbf{m}) . \end{aligned}$$

Hence

$$SSE_{r,H} - SSE_r = (\underset{\sim}{K}'\underset{\sim}{b}^o_r - \underset{\sim}{m})'[\underset{\sim}{K}'\underset{\sim}{GK} - \underset{\sim}{K}'\underset{\sim}{GP}(\underset{\sim}{P}'\underset{\sim}{GP})^{-1}\underset{\sim}{P}'\underset{\sim}{GK}]^{-1}(\underset{\sim}{K}'\underset{\sim}{b}^o_r - \underset{\sim}{m})$$

and

$$F(H_r) = \frac{SSE_{r,H} - SSE_r}{s\hat{\sigma}^2_r} \quad \text{with} \quad \hat{\sigma}^2_r = \frac{SSE_r}{N - r(\underset{\sim}{X}) + q} .$$

Timm and Carlson further reduce the symbolism by writing $\underset{\sim}{A} = \underset{\sim}{I} - \underset{\sim}{GP}(\underset{\sim}{P}'\underset{\sim}{GP})^{-1}\underset{\sim}{P}'$ and having the inverse matrix as $(\underset{\sim}{K}'\underset{\sim}{AGK})^{-1}$.

The differences in the notations used in the two references are seen in the following table.

Correspondence of Notation

	Searle	Timm and Carlson
Model	$\underset{\sim}{y} = \underset{\sim}{Xb} + \underset{\sim}{e}$	$\underset{\sim}{y} = \underset{\sim}{Wu} + \underset{\sim}{e}$
Inverse	$\underset{\sim}{G} = (\underset{\sim}{X}'\underset{\sim}{X})^-$	$\underset{\sim}{D}^{-1} = (\underset{\sim}{W}'\underset{\sim}{W})^{-1}$
Solution	$\underset{\sim}{b}^o = \underset{\sim}{GX}'\underset{\sim}{y}$	$\hat{\underset{\sim}{u}}_\Omega = \underset{\sim}{DW}'\underset{\sim}{y}$
Restrictions	$\underset{\sim}{P}'\underset{\sim}{b} = \underset{\sim}{\delta}$	$\underset{\sim}{R}'\underset{\sim}{u} = \underset{\sim}{\theta}$
Solution	$\underset{\sim}{b}^o_r$	$\hat{\underset{\sim}{u}}_{\tilde{\Omega}}$
Hypothesis	$H : \underset{\sim}{K}'\underset{\sim}{b} = \underset{\sim}{m}$	$H : \underset{\sim}{C}'\underset{\sim}{u} = \underset{\sim}{\xi}$
Solution	$\underset{\sim}{b}^o_{r,H}$	$\hat{\underset{\sim}{u}}_{\tilde{\omega}}$

References

Searle, S. R. [1971]. Linear Models, Wiley, New York.

Timm, N. H. and Carlson, J. E. [1973]. Analysis of variance through full rank models. Working Paper No. 21, Department of Educational Research, University of Pittsburgh.