HYPOTHESIS TESTING IN RESTRICTED LINEAR MODELS: CORRECTING AN ERROR

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Abstract

The numerator sum of squares in the F-statistic for testing a hypothesis in a linear model containing restrictions involving estimable functions is incorrect in Searle [1971]. The correction has been provided by Timm and Carlson [1973], for the full rank model. Details for the non-full rank model are given here.

The restricted linear model we consider is \( y = Xb + e \) subject to restrictions \( P'b = \delta \) where \( P'b \) is a set of \( r \) linearly independent estimable functions. For this model Searle [1971, p. 206] shows that a solution vector for \( b \) is

\[
b^0_r = b^0 - GP(P'GP)^{-1}(P'b^0 - \delta)
\]

(1)

where \( b^0 = GX'y \) and \( X'X = X'X \). The error sum of squares given on the same page is

\[
SSE_r = SSE + (P'b^0 - \delta)'(P'GP)^{-1}(P'b^0 - \delta)
\]

(2)

where \( SSE = y'[I - X(X'X)^{-1}]y \).

In testing the hypothesis \( H : K'b = m \) where \( K'b \) represents \( s \) linearly independent estimable functions, we confine attention to the case where the row spaces of \( P' \) and \( K' \) are linearly independent. On defining
The F-statistic for testing $H$ is then derived in Searle [1971] with numerator
\[ SSE_{r,H} - SSE = (Q'b^0 - \ell)'(Q'GQ)^{-1}(Q'b^0 - \ell). \]

This, as Timm and Carlson [1973, p. 33] point out, is wrong. It should be
\[ SSE_{r,H} - SSE = (Q'b^0 - \ell)'(Q'GQ)^{-1}(Q'b^0 - \ell) - (P'b^0 - \delta)'(P'GP)^{-1}(P'b^0 - \delta). \]

We show how this simplifies. From (3)
\[ \begin{bmatrix} P'GP \\ K'GP \\ K'GK \end{bmatrix}^{-1} \begin{bmatrix} P'GK \\ K'GP \\ K'GK \end{bmatrix}^{-1} = \begin{bmatrix} (P'GP)^{-1} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -(P'GP)^{-1}P'GK \\ 0 \end{bmatrix}W^{-1}[-K'GP(P'GP)^{-1} I] \]

with $W = K'GK - K'GP(P'GP)^{-1}P'GK$. Hence, using (3) again, (5) becomes
\[ SSE_{r,H} - SSE = \left( \begin{array}{c} (P'b^0 - \delta) \\ (K'b^0 - m) \end{array} \right)' \left( \begin{array}{c} (P'b^0 - \delta) \\ (K'b^0 - m) \end{array} \right) \]

\[ = W^{-1} \tau \text{ with } \tau = K'b^0 - m - K'GP(P'GP)^{-1}(P'b^0 - \delta) \]

\[ = \left( K'b^0 - m \right) \text{ from (2)} \]
Hence
\[ \text{SSE}_{r,H} - \text{SSE}_r = (K'\mathbf{b}^o_1 - \mathbf{m})'[K'GK - K'\mathbf{G}
\begin{bmatrix} \mathbf{P}' \mathbf{P} \end{bmatrix}^{-1} \mathbf{P}'GK]^{-1}(K'\mathbf{b}^o_1 - \mathbf{m}) \]
and
\[ F(H_r) = \frac{\text{SSE}_{r,H} - \text{SSE}_r}{\hat{\sigma}^2_r} \text{ with } \hat{\sigma}^2_r = \frac{\text{SSE}_r}{N - r(X) + q}. \]

Timm and Carlson further reduce the symbolism by writing
\[ A = \mathbf{I} - \mathbf{G}\mathbf{P} (\mathbf{P}'\mathbf{G})^{-1} \mathbf{P}' \]
and having the inverse matrix as \((K'\mathbf{A}GK)^{-1}\).

The differences in the notations used in the two references are seen in the following table.

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References
