A NOTE ON THE CONSTRUCTION OF ORTHOGONAL SERIES 1 BIB DESIGNS

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by

W. T. Federer and D. Raghavarao[1]

Cornell University and University of North Carolina at Chapel Hill

The orthogonal series (OS1) balanced incomplete block (BIB) designs with parameters

$$v = n^2,\ b = n(n + 1),\ r = n + 1,\ k = n,\ \lambda = 1 \qquad \ldots(1)$$

are known to exist when n is a prime or a prime power, and the construction is based on finite geometries or complete sets of mutually orthogonal latin squares (cf. Raghavarao [1971]). However, this series can also be constructed in an interesting way by using permutation matrices when n is a prime, and we describe this method in this note.

Let T_i be a $n \times n$ matrix with 1's in the i^{th} column and 0's elsewhere for $i = 1,2,\cdots,n$. Let $P_0, P_1, \cdots, P_{n-1}$ be n matrices of order $n \times n$ obtained by cyclical permutation of rows of I_n, the identity matrix of order n, as follows:

$$P_0 = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & \cdots & & \cdots & . \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix},\quad P_1 = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & \cdots & & \cdots & . \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}, \cdots,$$

$$P_{n-1} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ & \cdots & & \cdots & & . \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

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Now consider the $n^2 \times n(n+1)$ matrix N given by

$$N = \begin{bmatrix} T_1 & P_0 & P_1 & P_2 & \cdots & P_{n-1} \\ T_2 & P_1 & P_3 & P_5 & \cdots & P_{n-1} \\ T_3 & P_2 & P_5 & P_8 & \cdots & P_{n-1} \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ T_n & P_{n-1} & P_{n-1} & P_{n-1} & \cdots & P_{n-1} \end{bmatrix} \qquad \ldots(2)$$

where the subscripts of P are reduced to modulus n. One can easily verify that N is the well known incidence matrix of the BIB design with parameters (1). This method doesn't hold when n is a prime power or composite number.

REFERENCE

Raghavarao, D. [1971]. Constructions and Combinatorial Problems in Design of Experiments. Wiley, New York.

An Addendum

However, one need not restrict oneself to permutation matrices generated by a cyclical permutation of the rows of I_n. Instead consider a set of matrices P_i^*, $i = 0,1,\cdots,n-1$, where the n ones in the matrix appear in the matrix in the places where the i^{th} transversal of an $n \times n$ latin square would appear. Note that a latin square of side n with n transversals (i.e. an orthogonal pair of latin squares) exists for all n except n = 2 and 6. Thus to form the balanced incomplete block designs with parameters as given in (1), we proceed as follows to form N^*, the incidence matrix:

$$N^* = \left[\begin{array}{c|ccc|c} T_1 & & & & P_{n-1}^* \\ T_2 & & \text{(n-1) x (n-1) latin square of the } P_i^* \text{ for } i = 0,1,\cdots,n-2 & & P_{n-1}^* \\ \vdots & & & & \vdots \\ & & & & P_{n-1}^* \\ \hline T_n & P_{n-1}^* & \cdots & P_{n-1}^* & P_{n-1}^* \end{array}\right] \qquad \ldots(3)$$

The method of construction in (2) and (3) results in the incidence matrices of BIB designs for all n except n = 2 and 6. Also, note that (2) is a special case of (3) since a cyclical permutation of latin squares of side odd n results in a latin square orthogonal to the original one (cf. Hedayat and Federer[1969]).

An Example (n = 4)

In the following permutation matrices the ones appear in the places where the transversals of a latin square of order 4 would appear. Note that $NN' = 4I + J$ for the following design:

$v = n^2 = 16,\ b = 20,\ r = 5,\ \lambda = 1$

N =

1 0 0 0	1 0 0 0	0 1 0 0	0 0 1 0	0 0 0 1
1 0 0 0	0 1 0 0	1 0 0 0	0 0 0 1	0 0 1 0
1 0 0 0	0 0 1 0	0 0 0 1	1 0 0 0	0 1 0 0
1 0 0 0	0 0 0 1	0 0 1 0	0 1 0 0	1 0 0 0
0 1 0 0	0 1 0 0	0 0 1 0	1 0 0 0	0 0 0 1
0 1 0 0	1 0 0 0	0 0 0 1	0 1 0 0	0 0 1 0
0 1 0 0	0 0 0 1	1 0 0 0	0 0 1 0	0 1 0 0
0 1 0 0	0 0 1 0	0 1 0 0	0 0 0 1	1 0 0 0
0 0 1 0	0 0 1 0	1 0 0 0	0 1 0 0	0 0 0 1
0 0 1 0	0 0 0 1	0 1 0 0	1 0 0 0	0 0 1 0
0 0 1 0	1 0 0 0	0 0 1 0	0 0 0 1	0 1 0 0
0 0 1 0	0 1 0 0	0 0 0 1	0 0 1 0	1 0 0 0
0 0 0 1	0 0 0 1	0 0 0 1	0 0 0 1	0 0 0 1
0 0 0 1	0 0 1 0	0 0 1 0	0 0 1 0	0 0 1 0
0 0 0 1	0 1 0 0	0 1 0 0	0 1 0 0	0 1 0 0
0 0 0 1	1 0 0 0	1 0 0 0	1 0 0 0	1 0 0 0

REFERENCE

Hedayat, A. and Federer, W. T. [1969]. An application of group theory to the existence and nonexistence of orthogonal latin squares. Biometrika 56: 547-551.