ON SUM COMPOSITION OF FRACTIONAL FACTORIAL DESIGNS[1]

by

B. L. Raktoe[2]

Cornell University and University of Guelph

ABSTRACT

This paper shows how the direct sum operation can be utilized in obtaining from initial fractional factorial designs for two separate symmetrical factorials, a fractional factorial design for the corresponding asymmetric factorial.

1. INTRODUCTION

In design theory there are well known algebraic operations which lead to new designs when we start out from a set of initial designs. One of these operations, namely the direct product (or Kronecker product) operation, was utilized by Chakravarti [1956] to produce certain types of fractional factorial designs for the asymmetrical factorial. The designs developed by him through this method did not relate to arbitrary initial fractional factorial designs. These initial designs specifically arose from the existence of orthogonal arrays, which were much earlier shown to be quite useful in factorial design theory by Rao [1947].

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2] Visiting Professor of Biological Statistics, Cornell University, September 1, 1971 - August 31, 1972.

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To illustrate the direct product method we reproduce the following example, which follows immediately from theorem 1 of Chakravarti's [1956] paper. The orthogonal arrays D_1^* and D_2^* below are orthogonal main effect plans in $N_1 = 4$ and $N_2 = 9$ treatment combinations for the 2^3 and 3^4 factorials respectively.

D_1^*	D_2^*		
0 0 0	0 0 0 0	1 1 2 0	2 2 1 0
1 1 0	0 1 1 2	1 2 0 2	
1 0 1	0 2 2 1	2 0 2 2	
0 1 1	1 0 1 1	2 1 0 1	

The direct product design $D_1^* \otimes D_2^*$ in $N_1N_2 = 36$ treatment combinations provides orthogonal estimates of not only the main effects but also of the two factor interaction of one 2-level factor with one 3-level factor for the $2^3 \times 3^4$ factorial.

$D_1^* \otimes D_2^*$			
0 0 0 0 0 0 0	0 0 0 0 1 1 2		0 0 0 2 2 1 0
1 1 0 0 0 0 0	1 1 0 0 1 1 2	....	1 1 0 2 2 1 0
1 0 1 0 0 0 0	1 0 1 0 1 1 2		1 0 1 2 2 1 0
0 1 1 0 0 0 0	0 1 1 0 1 1 2		0 1 1 2 2 1 0

Besides the above designs there is a need to spell out the details of the direct product method for arbitrary initial designs and given arbitrary parameters under various assumptions on the total parametric vector. Such initial designs would encompass resolution III, IV and V designs. In some settings (especially when the orthogonality conditions are dropped) the resultant direct product design might be uneconomical from the viewpoint of number of treatment combinations. Thus, in the previous example, if main effects are the only ones of interest, it is clear that 36 treatment combinations are too many for estimation purposes. This is so because for the $2^2 \times 3^2$ we need only 7 treatment combinations to estimate the main effects non-orthogonally under the assumption that all interactions are zero. If it is desirable to have an estimate of the variance then clearly the number of observations should at least equal 8.

To obtain economical fractions we can resort to a different operation altogether, e.g. we can compose two initial designs using the direct sum operation. Before taking a formal approach consider the main effect design D_1 and D_2 consisting of $N_1 = 4$ and $N_2 = 6$ treatment combinations for the $k_1^{m_1} = 2^2$ and $k_2^{m_2} = 3^2$ factorials respectively:

$$\begin{array}{cc} \underline{D_1} & \underline{D_2} \\ 1\ 0 & 0\ 0 \\ 0\ 1 & 2\ 2 \\ 1\ 1 & 1\ 0 \\ 1\ 1 & 1\ 2 \\ & 0\ 1 \\ & 2\ 1 \end{array}$$

It is easily verified, that the design $D_1 \oplus D_2$ below

$$\underline{D_1 \oplus D_2} \qquad \begin{array}{cc:cc} 1 & 0 & & \\ 0 & 1 & & \\ 1 & 1 & \multicolumn{2}{c}{A_2} \\ 1 & 1 & & \\ \hdashline & & 0 & 0 \\ & & 2 & 2 \\ \multicolumn{2}{c:}{A_1} & 1 & 0 \\ & & 1 & 2 \\ & & 0 & 1 \\ & & 2 & 1 \end{array}, \text{ where } A_1 = \begin{array}{cc} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{array} \text{ and } A_2 = \begin{array}{cc} 2 & 0 \\ 2 & 0 \\ 2 & 0 \\ 2 & 0 \end{array}$$

is a non-singular main effect plan in $N_1 + N_2 = 4 + 6 = 10$ runs for the $k_1^{m_1} \times k_2^{m_2} = 2^2 \times 3^2$ asymmetrical factorial. The operation involved in producing this design is clearly a direct sum type of operation, which we will call compactly <u>sum composition</u>. It is clear that the crucial part in using this method is the specification of the matrices A_1 and A_2. The choice of these will depend on what kind of properties one wishes to impose on the resulting design, given certain properties on the initial designs.

In the next section we explore this new method in more detail and show how it always produces a design for an asymmetrical factorial $k_1^{m_1} \times k_2^{m_2}$ given the initial designs for the $k_1^{m_1}$ and $k_2^{m_2}$ factorials.

2. THE SUM COMPOSITION METHOD

Consider the $k_i^{m_i}$ factorial and suppose that the experimenter partitions the $k_i^{m_i} \times 1$ parametric vector β_i as $\beta_i' = (\beta_{i1}' \vdots \beta_{i2}' \vdots \beta_{i3}')$, where β_{i1} is the $p_{i1} \times 1$ vector of parameters to be estimated, β_{i2} is the $p_{i2} \times 1$ vector of parameters not of interest and not assumed to be zero, and, β_{i3} is the $(k_i^{m_i} - p_{i1} - p_{i2}) \times 1$ vector of parameters assumed to be zero. We assume the first element of both β_{11} and β_{21} in respectively the $k_1^{m_1}$ and $k_2^{m_2}$ factorials to be equal to the mean μ. Also, we limit ourselves in this paper to the most popular case, i.e. the case where $p_{12} = p_{22} = 0$. Let D_i, $i = 1,2$, be a design consisting of N_i treatment combinations from the $k_i^{m_i}$ factorial such that the vector β_{i1} is estimable. Consider the design

$$\begin{array}{c|c} D_1 & A_2 \\ \hline A_1 & D_2 \end{array}$$

where A_1 is $N_2 \times m_1$ and A_2 is $N_1 \times m_2$, and, the rows of A_i are treatment combinations from the $k_i^{m_i}$ factorial. We desire a choice of A_1 and A_2 such that the resulting design $D_1 \oplus D_2$ provides unbiased estimates for the elements of the vector $\beta_{11} \cup \beta_{21}$. (Here $\beta_{11} \cup \beta_{21}$ is a $(p_{11} + p_{21} - 1) \times 1$ vector whose entires are elements of the union of β_{11} and β_{21} when these are considered as sets.) Such a design $D_1 \oplus D_2$ consisting of $N_1 + N_2$ treatment combinations to estimate $p_{11} + p_{21} - 1$ parameters is said to be obtained using the sum composition of D_1 and D_2.

We show that when A_2 consists of N_1 repetitions of any arbitrary treatment combination of the $k_2^{n_2}$ factorial and A_1 consists of N_2 repetitions of any treatment combination of D_1, then $D_1 \oplus D_2$ is such that the rank of its design matrix is equal to $p_{11} + p_{21} - 1$. This means that given any two non-singular designs D_1 and D_2 from the $k_1^{m_1}$ and $k_2^{m_2}$ factorials respectively such that β_{11} and β_{21} are estimable, then one may always obtain a non-singular design $D_1 \oplus D_2$ from the $k_1^{m_1} \times k_2^{m_2}$ factorial such that $\beta_{11} \cup \beta_{21}$ is estimable.

The proof of this proposition follows by noting that the design matrix $X_{D_1 \oplus D_2}$ of the design $D_1 \oplus D_2$ is essentially of the form:

$$X_{D_1 \oplus D_2} = \left[\begin{array}{c:c:c} 1_{N_1} & X_{D_1} & X_{A_2} \\ \hdashline 1_{N_2} & X_{A_1} & X_{D_2} \end{array}\right]_{(N_1+N_2) \times (p_{11}+p_{21}-1)}$$

where 1_{N_i} is column vactor of order N_i and X_{D_i} is the design matrix determined by D_i alone with respect to β_{i1} with the element μ deleted. If the h^{th} treatment combination of D_1 is repeated in A_1, then clearly $X_{A_1} = 1_{N_2} \otimes (h^{th}$ row of $X_{D_1})$.

Also, if the design vector of the repeated arbitrary treatment combination in A_2 is denoted by x'_{A_2}, then $X_{A_2} = 1_{N_1} \otimes x'_{A_2}$. Using row and column operations one sees that the matrix $X_{D_1 \oplus D_2}$ is rank equivalent to the matrix:

$$
\tilde{X}_{D_1 \oplus D_2} = \left[\begin{array}{c|c|c}
1 & & 0\ 0\ \cdots\ 0 \\
0 & \tilde{X}_{D_1} & 0\ 0\ \cdots\ 0 \\
\vdots & & \cdots\cdots \\
0 & & 0\ 0\ \cdots\ 0 \\
\hline
0 & 0\ 0\ \cdots\ 0 & \\
0 & 0\ 0\ \cdots\ 0 & \tilde{X}_{D_2} \\
\vdots & \cdots\cdots & \\
0 & 0\ 0\ \cdots\ 0 &
\end{array}\right]
$$

where the first row of $\tilde{X}_{D_1}$ consists entirely of 0's. Hence the rank of $X_{D_1 \oplus D_2}$ is equal to $1 + (p_{11}-1) + (p_{21}-1) = p_{11} + p_{21} - 1$.

3. REMARKS

Hedayat and Seiden [1971] among other results on sum composition of latin squares, have obtained a result which equivalently can be stated as follows:

Under certain regularity conditions, the existence of an orthogonal resolution III plan consisting of k_1^2 treatment combinations for 4 factors each at k_1 levels and an orthogonal resolution III plan consisting of k_2^2 treatment combinations for 4 factors each at k_2 levels implies the existence of an orthogonal resolution III plan consisting of $k_1^2 + k_2^2 + 2k_1k_2$ points for 4 factors each at k_1+k_2 levels. This type of fraction in this higly specialized setting, falls somewhere between the direct product type of design and the sum composition type of design.

4. FURTHER RESEARCH

The notions of the previous section suggests the following problems:

(a) Are there other sufficient conditions and necessary conditions on A_1 and A_2 such that $\beta_{11} \cup \beta_{21}$ is estimable?

(b) What is the generalization of the sum composition method to the $k_1^{m_1} \times k_2^{m_2} \times \cdots \times k_t^{m_t}$ factorial?

(c) How do the concepts of orthogonality and balance relate to the design produced by the sum composition method, given that the initial designs possess the properties?

(d) How do we apply the sum composition method given that the initial designs are of resolution III, IV or V?

(e) How do we guarantee optimality of the design produced by the sum composition given that the initial designs are optimal in some sense?

(f) How does the permutation theory as expanded in Srivastava, Raktoe and Pesotan [1971] apply to the sum composition method?

(g) How can we reduce the number of treatment combinations in the composed design to retain estimability of $\beta_{11} \cup \beta_{21}$?

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