PARTIALLY BALANCED DESIGNS AND PROPERTIES A AND B

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ABSTRACT

The following relations have been demonstrated in this paper: an incomplete block design with property A in the Kurkjian-Zelen sense, is a balanced factorial experiment in the Shah sense which in turn is a partially balanced incomplete block design having a "binary number association scheme" and conversely; likewise, a rectangular design which has property A in columns and property B in rows in the Zelen-Federer sense, is a balanced factorial rectangular experiment which in turn is a partially balanced rectangular design, and conversely. The establishment of these relations is useful in constructing block and row-column designs which are partially balanced and in constructing fractional replicates which are balanced.

1. INTRODUCTION AND SUMMARY

Kurkjian and Zelen [1963] introduced a structural property, which was designated as Property A, of the (block) incidence matrix N associated with a class of block and direct product experiment designs. Extending this idea, Zelen and Federer [1964] introduced two structural properties, which were
designated as Property A and Property B, associated with experiment designs for two-way elimination of heterogeneity, such as a k-row by b-column rectangular experiment design, and direct product designs. Property A is associated with the column (block) incidence matrix $N$ and Property B is associated with the row incidence matrix $\bar{N}$. In the paper by Paik and Federer [1971], the incomplete block designs (IBD) possessing Property A were designated as PA type incomplete block designs and likewise, those two-way elimination of heterogeneity designs possessing both Property A and Property B were designated as PAB type designs. Also, these authors showed that, under factorial structure, PA type incomplete block designs are "balanced factorial experiments (BFE)", as defined by Shah [1960], and are partially balanced incomplete block (PBIB) designs, with relevant parameters.

In this paper, we show that every BFE is a PA type incomplete block design but a PBIB design is not always a balanced factorial incomplete block design (BFIBD) nor a PA type incomplete block design. We define a "binary number association scheme (BNAS)" and obtain a condition for which a PBIB design is a BFIBD or a PA type IBD. Also, we define n-ary partially balanced block (NPBB) designs which may have a specified treatment appearing an arbitrary number of times in a block, and discuss some properties of NPBB designs.

The relationships discussed herein may be summarized as follows:

1. $(PA \text{ and } PB) \nRightarrow PAB$
2. $BFE \nRightarrow PA \nRightarrow (NPBB \text{ with BNAS})$
3. $PBIB \text{ without BNAS is not a BFE}$
4. $PBIB \nRightarrow NPBB \text{ but not conversely}$
5. $BFIBD \nRightarrow (PBIB \text{ with BNAS})$
6. $(PA \text{ type IBD}) \nRightarrow (PBIB \text{ with BNAS})$
7. $PAB \nRightarrow (NPBB \text{ in rows and in columns})$
8. $PAB \nRightarrow \text{balanced factorial rectangular experiment (BFRE), but not conversely.}$
2. PRELIMINARIES

The following notation (after Kurkjian and Zelen [1963]) will be used:

1_{m_i} : m_i \times 1 column vector having all elements unity,

J_{m_i} = 1_{m_i} 1_{m_i}' : m_i \times m_i matrix with all elements unity,

I_{m_i} : m_i \times m_i identity matrix,

\begin{align*}
\mathbf{1}_{m_i} & \begin{cases} 
1_{m_i} & \text{if } x_i = 0 \\
I_{m_i} & \text{if } x_i = 1 
\end{cases} \\
\mathbf{J}_{m_i} & \begin{cases} 
I_{m_i} & \text{if } \delta_i = 0 \\
J_{m_i} & \text{if } \delta_i = 1 
\end{cases} \\
\mathbf{M}_i & \begin{cases} 
1_{m_i} & \text{if } x_i = 0 \\
M_i & \text{if } x_i = 1 
\end{cases} \\
\mathbf{M}_{i}^{X_i} & \begin{cases} 
1_{m_i} & \text{if } x_i = 0 \\
M_i & \text{if } x_i = 1 
\end{cases}
\end{align*}

The direct product, or Kronecker product, of \( \mathbf{M}_i^{X_i} \) and \( \mathbf{M}_j^{X_j} \) will be written as \( \mathbf{M}_i^{X_i} \otimes \mathbf{M}_j^{X_j} \) and in general, the joint direct product of \( \mathbf{M}_i^{X_i} \) \((i=1,2,\ldots,n)\) will be written as \( \prod_{i=1}^{n} \mathbf{M}_i^{X_i} \).

Let there be \( v \) treatments, each replicated \( r \) times in \( b \) blocks of \( k \) plots each. Let \( N = ||n_{ij}||, i=1,2,\ldots,v; j=1,2,\ldots,b, \) be the incidence matrix of the design, where \( n_{ij}(=0,1,2,\ldots) \) is equal to the number of times the \( i^{th} \) treatment occurs in the \( j^{th} \) block. The model assumed is

\begin{equation}
\begin{align*}
y_{ij} & = \mu + t_i + b_j + \epsilon_{ij},
\end{align*}
\end{equation}

where \( y_{ij} \) is the yield of the plot in the \( j^{th} \) block to which the \( i^{th} \) treatment is applied, \( \mu \) is the overall effect, \( t_i \) is the effect of the \( i^{th} \) treatment,
b_j is the effect of the jth block, and \( \epsilon_{ij} \) is the experimental error. The effects \( \mu, t_i, b_j \), are assumed to be fixed constants, while the error \( \epsilon_{ij} \)'s are assumed to be independent normal variates with mean zero and variance \( \sigma^2 \). Let \( T_i \) be the total yield of all plots having the ith treatment, \( B_j \) be the total yield of all the plots of the jth block and \( \hat{t}_i \) be a solution for \( t_i \) in the normal equations. Further, denote the column vectors \((T_1, T_2, \ldots, T_v)'\), \((B_1, B_2, \ldots, B_b)'\), \((t_1, t_2, \ldots, t_v)'\), and \((\hat{t}_1, \hat{t}_2, \ldots, \hat{t}_v)'\) by \( T \), \( B \), \( t \) and \( \hat{t} \) respectively. It is known that the reduced normal equations for intra-block estimates of treatment effects are

\[
(2.2) \quad C \hat{t} = Q,
\]

where

\[
(2.3) \quad C = rI_v - \frac{1}{k} NN', \quad \text{and}
\]

\[
(2.4) \quad Q = T - \frac{1}{k} NB.
\]

The matrix \( C \) defined in (2.3) will be called the C-matrix of the design. The solution of (2.2) is \( \hat{t} = C^+ Q \), where \( C^+ \) is a generalized inverse of \( C \).

Consider a factorial experiment with \( n \) factors \( F_1, F_2, \ldots, F_n \), where factor \( F_i \) has \( m_i \) levels for \( i = 1, 2, \ldots, n \), there being \( v = \sum_{i=1}^{n} m_i \) treatments. Kurkjian and Zelen [1963] introduced a structural property of the design which was related to the block (or column) incidence matrix \( N \) of the design. This structural property was termed Property A and may be defined as follows: A block design will be said to have Property A, or will be called a PA type block design, if

\[
(A) \quad NN' = \sum_{s=0}^{n} \left\{ \sum_{\delta_1+\delta_2+\ldots+\delta_n = s} h(\delta_1, \delta_2, \ldots, \delta_n) \prod_{i=1}^{n} \delta_i \right\}
\]

where \( \delta_i = 0 \) or 1 for \( i = 1, 2, \ldots, n \), and \( h(\delta_1, \delta_2, \ldots, \delta_n) \) are constants. In this case, we obtain the following solution for equation (2.2):

\[
(2.2) \quad C \hat{t} = Q.
\]
(2.5) \[ \hat{t} = \sum_{s=1}^{n} \left\{ \left( - \frac{n}{\sum_{i=1}^{n} \Theta_i^{N_{i-1}^{*}}} \right) \frac{\sum_{i=1}^{n} x_i^{N_{i-1}^{*}}}{rv\Theta(x_1,x_2,\ldots,x_n)} \right\} q. \]

where \( \Theta(x_1,x_2,\ldots,x_n) \) is the efficiency factor associated with the estimate of generalized interaction \( F_1^{x_1} F_2^{x_2} \ldots F_n^{x_n} \) and

(2.6) \[ r\Theta(x_1,x_2,\ldots,x_n) \]

\[ = \sum_{s=0}^{n-1} \left\{ \left( - \frac{n}{\sum_{i=1}^{n} \Theta_i^{N_{i-1}^{*}}} \right) \frac{\sum_{i=1}^{n} (1-x_i) \Theta_i^{N_{i-1}^{*}}}{v} \right\} \]

for \( g(0,0,\ldots,0) = r - \frac{1}{k} g(0,0,\ldots,0) \), \( g(\delta_1,\delta_2,\ldots,\delta_n) = -\frac{1}{k} \Theta(\delta_1,\delta_2,\ldots,\delta_n) \) if \( (\delta_1,\delta_2,\ldots,\delta_n) \neq 0 \).

In a \( k \times b \) rectangular experiment design with \( v = \sum_{i=1}^{n} m_i \) treatments being replicated \( r \) times each, suppose the column incidence matrix \( N \) has Property A and the row incidence matrix \( \hat{N} \) of the experiment design has Property B as introduced by Zelen and Federer [1964], i.e.,

(B) \[ \hat{N}^N = \sum_{s=0}^{n} \left\{ \left( - \frac{n}{\sum_{i=1}^{n} \Theta_i^{N_{i-1}^{*}}} \right) \frac{\sum_{i=1}^{n} \Theta_i^{N_{i-1}^{*}}}{v} \right\} , \]

where \( \hat{N}(\delta_1,\delta_2,\ldots,\delta_n) \) are constants, then the \( k \times b \) rectangular design will be said to have Property A and Property B or will be called a PAB type \( k \times b \) rectangular design. It is known that, in a \( k \times b \) rectangular design with \( v \) treatments being replicated \( r \) times each, the reduced normal equations for estimating the treatment effects may be written as
where

\[ \hat{C} = rI_v - \frac{1}{k}NN' - \frac{1}{b}NN' + J_v(r/v), \]

and

\[ \hat{Q} = T - \frac{1}{k}NR - \frac{1}{b}NR + I_v(g/v), \]

where \( R = (R_1, R_2, \ldots, R_k)' \), \( R_s = \) total yield of all the plots of the \( s^{th} \)
row; \( g = \) total yield of all the plots in the experiment. The solution of
(2.7) is \( \hat{t} = \hat{C}^{-1}\hat{Q} \), where \( \hat{C}^{-1} \) is a generalized inverse of matrix \( \hat{C} \). In the PAB

\[ \hat{C}^{-1} = \sum_{s=1}^{n} \left\{ \Sigma \frac{n}{rv^s(x_1, x_2, \ldots, x_n)} \right\} \]

where

\[ n \]

\[ \sum_{s=0}^{n-1} \left\{ \Sigma \frac{n}{rv^s(x_1, x_2, \ldots, x_n)} \right\} \]

for \( \tilde{g}(0,0,\ldots,0) = r - \frac{1}{k}n(0,0,\ldots,0) - \frac{1}{b}m(0,0,\ldots,0), \)
\( \tilde{g}(\delta_1, \delta_2, \ldots, \delta_n) \)
\( = \frac{1}{k}n(\delta_1, \delta_2, \ldots, \delta_n) - \frac{1}{b}m(\delta_1, \delta_2, \ldots, \delta_n) \) if \( (\delta_1, \delta_2, \ldots, \delta_n) \neq 0 \).

Paik and Federer [1971] have proved that PA type incomplete block
designs are BFE and PAB type rectangular designs achieve "complete balance"
over each of the interactions in the sense of the following definitions
given by Shah [1960]:

**Definition 2.1.** (Shah). "Complete balance" is achieved over an interaction
if and only if all the normalized contrasts belonging to the same interaction
are estimated with the same variance.
Definition 2.2. (Shah). An experiment will be called a balanced factorial experiment (BFE), if the following conditions are satisfied:

(a) Each of the treatments is replicated the same number of times.
(b) Each of the blocks has the same number of plots.
(c) Estimates of contrasts belonging to different interactions are uncorrelated with each other.
(d) "Complete balance" is achieved over each of the interactions.

Shah [1960] proved that a BFE is a PBIB design with relevant parameters. Therefore, we state that a **PA type block design is a BFE and is a PBIB design** with relevant parameters. Also, a **PAB type rectangular design is a balanced factorial rectangular experiment (BFRE)**.

3. BALANCED FACTORIAL EXPERIMENTS AND PA TYPE BLOCK DESIGNS

We shall prove that every BFE is a PA type block design. Consider a BFE in **n** factors, \(F_1, F_2, \ldots, F_n\) at \(m_1, m_2, \ldots, m_n\) levels respectively, with each treatment combination replicated \(r\) times in \(b\) blocks of \(k\) plots each, there being \(v = \prod_{i=1}^{n} m_i\) treatments. Let \(L = \left[q_1, q_2, \ldots, q_n\right]\), where \(q_i = 0\) or \(1\) for \(i = 1, 2, \ldots, n\), be a \(v \times (m_i-1)q_i\) matrix formed by a complete set of \(\pi (m_i-1)q_i\) normalized orthogonal vectors forming the generalized interaction \(F_{i_1}F_{i_2}\ldots F_{i_n}\), then using the results of Shah [1958, 1960] and Paik and Federer [1971], the **C-matrix defined in (2.2) is uniquely represented and given by**

\[
C = r \sum_{s=1}^{n} \left\{ \delta(q_1, q_2, \ldots, q_n)Lq_1q_2q_3 \ldots q_n \right\},
\]

which can also be written as
where the $f_{ij}(q_1, q_2, \ldots, q_n)$ are the elements of $L_1 q_1 \ldots q_n L_1' q_2 \ldots q_n$ corresponding to the $i$th row and $j$th column. Under the assumed model (2.1), the variance-covariance matrix of a contrast set corresponding to the generalized interaction $F_1 \cdot F_2 \cdot \ldots \cdot F_n$ will be

$$
\sigma^2 / \theta(q_1, q_2, \ldots, q_n) I_n
$$

and since representation of the matrix $C$ is unique, i.e., if $L_1 q_1 \ldots q_n$ and $L^* q_1 \ldots q_n$ are two matrices formed by two sets of \( \prod_{i=1}^{n} (m_i - 1) q_i \) normalized orthogonal vectors forming the generalized interaction $F_1 \cdot F_2 \cdot \ldots \cdot F_n$, then

$$
L_1 q_1 \ldots q_n L_1' q_2 \ldots q_n = L^* q_1 \ldots q_n L^*_1 q_2 \ldots q_n.
$$

Define $P_i$ to be an $m_i \times (m_i - 1)$ matrix such that $P_i P_i' = I_{m_i - 1}$ and $1_m P_i = 0$ for $i = 1, 2, \ldots, n$, and define, for $i = 1, 2, \ldots, n$,

$$
p_i^{q_i} = \begin{cases} 
\frac{1}{\sqrt{m_i}}, & \text{if } q_i = 0 \\
1_{m_i}, & \text{if } q_i = 1
\end{cases}
$$

Let

$$
L^* q_1 q_2 \ldots q_n = \prod_{i=1}^{n} \Theta^{q_i}.
$$
then $L^*_q q_1 q_2 \cdots q_n$ is a $n \times \prod_{i=1}^{n} (m_i-1)^{d_i}$ matrix formed by a complete set of

$\prod_{i=1}^{n} (m_i-1)^{d_i}$ normalized orthogonal vectors and we obtain:

$$L^*_q q_1 q_2 \cdots q_n q_3 q_4 \cdots q_n = \left( \prod_{i=1}^{n} \otimes q_i \right) \left( \prod_{i=1}^{n} \otimes q_i \right)'$$

$$= \sum_{i=1}^{n} \otimes q_i (P^q_i)' ,$$

and from (3.5)

$$p^q_i (P^q_i)' = \begin{cases} \frac{1}{m_i} J_{m_i}, & \text{if } q_i = 0 \\ \frac{1}{m_i} J_{m_i} - \frac{1}{m_i} J_{m_i}, & \text{if } q_i = 1 . \end{cases}$$

Now, from (3.1), (3.4), (3.7) and (2.3), we obtain:

$$NN' = \text{kr} \left( \sum_{i=1}^{n} \otimes \mathbb{I}_{m_i} - \sum_{i=1}^{n} \sum_{s=1}^{\otimes q_i + q_s + \cdots + q_n = s} \theta(q_1, q_2, \ldots, q_n) \right)$$

$$= \prod_{i=1}^{n} \otimes q_i (P^q_i)''' ,$$

and from (3.8), we know that the incidence matrix $N$ has Property A.

**Theorem 3.1.** Every BFE is a PA type block design and conversely.

**Example 3.1.** In a BFE with $v = m_1 m_2$,

$$c = r \sum_{s=1}^{2} \left\{ \sum_{q_1 + q_2 = s} \theta(q_1, q_2) \prod_{i=1}^{2} p^0_i (P^0_i)' \right\}$$

$$= r \left\{ \theta(0,1)P^0_1 (P^0_1)' + \theta(1,0)P^0_1 P^0_2 (P^0_2)' + \theta(1,1)P^0_1 P^0_2 (P^0_2)' \right\}$$
= r\left( e(0,1) \left( \frac{1}{m_1m_2} \otimes m_2 - \frac{1}{m_1m_2} m_1 m_2 \right) + e(1,0) \left( \frac{1}{m_1m_2} \otimes m_2 - \frac{1}{m_1m_2} m_1 m_2 \right) \right) \\
+ e(1,1) \left( I_{m_1} - \frac{1}{m_1m_2} \otimes I_{m_2} - \frac{1}{m_1m_2} m_1 m_2 \right) \\
= r e(1,1) I_{m_1m_2} + r \left( \frac{e(1,0)}{m_2} - \frac{e(1,1)}{m_2} \right) I_{m_1} \otimes m_2 + r \left( \frac{e(0,1)}{m_1} - \frac{e(1,1)}{m_1} \right) I_{m_1} \otimes I_{m_2} \\
+ r \left( \frac{e(1,1)}{m_1m_2} - \frac{e(0,1)}{m_1m_2} - \frac{e(1,0)}{m_1m_2} \right) I_{m_1m_2}.

Hence, \( g(0,0) = r e(1,1) \), \( g(0,1) = r \left( \frac{e(1,0)}{m_2} - \frac{e(1,1)}{m_2} \right) \), and \( g(1,0) = r \left( \frac{e(0,1)}{m_1} - \frac{e(1,1)}{m_1} \right) \).

Then, we obtain:

\[ r e(1,1) = g(0,0), \quad r e(1,0) = g(0,0) + m_2 g(0,1), \]

and

\[ r e(0,1) = g(0,0) + m_1 g(1,0). \]

4. BALANCED FACTORIAL EXPERIMENTS AND PBIB DESIGNS

Consider a BFE with \( n \) factors as described in section 3. Two treatments are \( p_1p_2\ldots p_n \)th associates, where \( p_i = 1 \), if the \( i \)th factor occurs at the same level in both the treatments and \( p_i = 0 \) otherwise; \( p_1p_2\ldots p_n \) will denote the number of times these treatments occur together in a block.

Then we have

\[ n_{p_1p_2\ldots p_n} = \sum_{s=0}^{n} \left( \frac{\Sigma}{p_1 + p_2 + \ldots + p_n = s} \pi (m_1-1-p_1) \right)^n. \]
This association is called the "binary number association scheme (BNAS)."

Note: The above association scheme was defined by Shah [1960], but the association scheme was not named. Also, the association scheme could be called the "factorial association scheme" but Srivastava and Anderson [1971] have used the name "factorial association scheme" in Multidimensional Partially Balanced designs in a more general sense.

In the above BFE, suppose that model (2.1) is assumed. Using (2.3) and (3.2), let

\[
\tag{4.2} C = rI_v - \frac{1}{k}\mathbf{N}' = r\left\{\sum_{s=1}^{n} \Theta(q_1, q_2, \ldots, q_n) f_{ij}(q_1, q_2, \ldots, q_n)\right\}, \quad i, j = 1, 2, \ldots, v,
\]

where \( f_{ij}(q_1, q_2, \ldots, q_n) \) is the element of \( L_q^* L_q^* \) corresponding to the \( i \)th row and \( j \)th column. Shah [1958] showed that \( f_{ij}(q_1, q_2, \ldots, q_n) \) depends only on the associates, say \( p_1 p_2 \ldots p_n \), between the \( i \)th and \( j \)th treatment. Let us denote this by \( f^q_{p} \), where \( p = \sum_{h=1}^{n} p_h 2^{n-h} \), \( q = \sum_{h=1}^{n} q_h 2^{n-h} \), \( p = m = 2^n \) denotes all levels equal (\( i = j \)), and \( f^q_{m} \) is a diagonal element.

Then, from (4.2)

\[
\sum_{q=1}^{m} \theta_q f^q_m = r - \frac{r}{k}, \quad \text{where} \quad \Theta_q = \Theta(q_1, q_2, \ldots, q_n) \quad \text{such that} \quad \Theta_q = \Theta(q_1, q_2, \ldots, q_n) \quad \text{such that} \quad \Theta_q = \Theta(q_1, q_2, \ldots, q_n) \quad \text{such that} \quad \Theta_q = \Theta(q_1, q_2, \ldots, q_n)
\]

\[
\sum_{q=1}^{m} \theta_q f^q_p = - \frac{\lambda_p}{k}, \quad p \neq m,
\]

\[
\tag{4.3}
\sum_{q=1}^{m} \theta_q f^q_p = - \frac{\lambda_p}{k}, \quad p \neq m,
\]
where \( \lambda_p = \lambda_{P_1}P_2 \cdots P_n \) such that \( p = \sum_{h=1}^{n} \lambda_p \) \( 2^{n-h} \) and \( \lambda_p \) equals the number of times two \( p \) th associate treatments occur together.

From (4.3), we obtain the following expression:

\[
\begin{bmatrix}
 f_0^0 & f_1^0 & \cdots & f_m^0 \\
 f_0^1 & f_1^1 & \cdots & f_m^1 \\
 \vdots & \vdots & \ddots & \vdots \\
 f_0^m & f_1^m & \cdots & f_m^m
\end{bmatrix}
\begin{bmatrix}
 \theta_0 \\
 \theta_1 \\
 \vdots \\
 \theta_m
\end{bmatrix}
= \begin{bmatrix}
 \lambda_0 \\
 \lambda_1 \\
 \vdots \\
 \lambda_m
\end{bmatrix}
\]

(4.4)

where \( \lambda_m = r - r_k \), \( f_p^0 = \frac{1}{v} \) for \( p = 0, 1, \ldots, m \), and \( \theta_0 \) is a dummy parameter always equal to zero, introduced to simplify the inverse relation. Shah [1958] also obtained the following relation for (4.4):

\[
r \prod_{i=1}^{n} \Phi_i(1) \cdot \theta = - \frac{1}{k} \lambda
\]

(4.5)

where \( \Phi_i(1) = \frac{1}{m_i} \left( \frac{1}{m_i} \right) \), \( \theta = (\theta_0, \theta_1, \ldots, \theta_m)' \), and \( \lambda = (\lambda_0, \lambda_1, \ldots, \lambda_m)' \).

Since \( \theta_0 = 0 \), \( \theta_m = r - r_k \), equation (4.4) can be written as:

\[
\begin{bmatrix}
 f_0^0 & f_1^0 & \cdots & f_m^0 \\
 f_0^1 & f_1^1 & \cdots & f_m^1 \\
 \vdots & \vdots & \ddots & \vdots \\
 f_0^m & f_1^m & \cdots & f_m^m
\end{bmatrix}
\begin{bmatrix}
 \theta_1 \\
 \theta_2 \\
 \vdots \\
 \theta_m
\end{bmatrix}
= \begin{bmatrix}
 \lambda_0 \\
 \lambda_1 \\
 \vdots \\
 \lambda_m
\end{bmatrix}
\]

(4.6)

For convenience, let us denote this as follows:

\[
rF^* \theta = - \frac{1}{k} \lambda
\]
Consider a PBIB design having parameters $v = \frac{n}{\pi} m_1, b, r, k,$ and $\lambda_0, \lambda_1, \ldots, \lambda_m,$ then we may obtain equation (4.6) for the unknown variables $\theta_1, \theta_2, \ldots, \theta_m.$ Suppose

\[(4.7) \quad \rho(F^*) = \rho(F^*, \lambda) = m,\]

where $\rho(F^*)$ denotes the rank of the matrix $F^*,$ then we may obtain a unique solution for $\theta^+.$ Furthermore, if the above PBIB design is connected, none of $\theta_i,$ $i = 1, 2, \ldots, m,$ are zero. Hence, the above PBIB design is a balanced factorial incomplete block design (BFIBD) and is a PA type incomplete block design. Now, we proceed to prove relation (4.7). That $\rho(F^*) = m,$ is clear from (4.4).

Next, since $r_p^q$ only depends upon the $p^{th}$ associates between two treatments in the $L_{q} L_{q}',$ where $p$ replaces $(p_1, p_2, \ldots, p_n)$ and $q$ replaces $(q_1, q_2, \ldots, q_n)$ such that $p = \sum_{h=1}^{n} p_h 2^{n-h}$ and $q = \sum_{h=1}^{n} q_h 2^{n-h}$ respectively, the number of $r_p^q$ in any row or column of the $L_{q} L_{q}'$ is $p_1 p_2 \cdots p_n$ and since $i_{v}(L_{q} L_{q}') = 0,$ we have

\[(4.8) \quad \sum_{p=0}^{m} n_p r_p^q = 0 \quad \text{for } q = 1, 2, \ldots, m,
\]
or

\[(4.9) \quad \sum_{p=0}^{m-1} n_p r_p^q = - \frac{r_q^q}{m} \quad \text{for } q = 1, 2, \ldots, m.
\]

Also, it is known that

\[(4.10) \quad \sum_{p=0}^{m-1} n_p \lambda_p = r-rk.\]
Thus, from (4.9) and (4.10), we have proved that \( p(F^*) = o(F^*, k) \).

Theorem 4.1. Every BFIBD is a PBIB with BNAS and conversely. Also, every PA type IBD is a PBIB with BNAS and conversely.

In a PBIB design with \( v = m \), each of the \( \lambda_{q_1 q_2 \ldots q_n} \) for all \( q_1 = 0 \) or 1, does not always have a unique value, i.e., a PBIB design does not always have a "binary number association scheme".

Example 4.1. \( v = 2 \times 5 \), \( r = 2 \), \( k = 4 \), \( b = 5 \).

\[
\begin{array}{ccccc}
 b_1 & b_2 & b_3 & b_4 & b_5 \\
(00) & (04) & (12) & (14) & (03) \\
(01) & (10) & (13) & (02) & (11) \\
(02) & (11) & (01) & (10) & (13) \\
(03) & (00) & (04) & (12) & (14) \\
\end{array}
\]

This design is a triangular PBIB design with two associate classes. However, if we consider the BNAS, we may find \( \lambda_{00} = 1 \) or 0, \( \lambda_{01} = 1 \), and \( \lambda_{11} = 6 \), so the above PBIB design is not a BFE.

5. \textit{n-ary PARTIALLY BALANCED BLOCK DESIGNS (NPBB DESIGNS)}

In the classical BIB design or PBIB design, no treatment appears more than once in a block. However, we may wish to apply some treatments more than once in a block. In such cases, the following definition of \textit{n-ary partially balanced block designs} (NPBB (see Tocher [1952])) may be useful for application.

Definition 5.1. (i). The experimental material is divided into \( b \) blocks of \( k \) units each, some treatments may appear \( n_{ij} = 0, 1, 2, \ldots, n-1 = \) a specified number, times in the same block.

(ii). All treatments will be replicated the same number (say \( r \)) of times in the same number (say \( q \)) of blocks.
(iii). There can be established a relation of association between any two treatments satisfying the following requirements.

(a). Two treatments are either first associates, second associates, ..., or \( u \) th associates.

(b). Each treatment has exactly \( n_i \) \( i \) th associates (\( i = 1, 2, ..., u \)).

(c). Given any two treatments which are \( i \) th associates, the number of treatments common to the \( j \) th associate of the first and \( k \) th associate of the second is \( p_{jk}^i \) and is independent of the pair of treatments we start with. Also \( p_{jk}^i = p_{kj}^i \) (\( i, j, k = 1, 2, ..., u \)).

(iv). Two treatments which are \( i \) th associates occur together the same number (say \( \lambda_i \)) of times. For example, if two treatments, \( t \) and \( t' \) say, are \( i \) th associates and they are replicated \( r_{tj} \) and \( r_{t'j} \) times in the \( j \) th block respectively, and \( r = \sum \limits_{j=1}^{b} r_{tj} \) for \( t = 1, 2, ..., v \), then

\[
\lambda_i = \sum_{j=1}^{b} r_{tj} r_{t'j}
\]

is constant as long as two treatments \( t \) and \( t' \) are \( i \) th associates. If \( \lambda_1 = \lambda_2 = \ldots = \lambda_u = \text{constant} \), the design may be called an \( n \)-ary balanced block design (NBB design).

(v). \( r^* = \sum \limits_{j=1}^{b} r_{tj}^2 \) is constant for \( t = 1, 2, ..., v \).

The following relation between the parameters will hold:

(1). \( vr = bk \)

(2). \( n_1 + n_2 + \ldots + n_u = v - 1 \).

Suppose that the \( t \) th treatment is replicated \( r_{tj} \) times in the \( j \) th block, then the \( t \) th treatment appears together with others \( r_{tj} (k - r_{tj}) \) times in the same block, and \( \lambda_i = \sum \limits_{j=1}^{b} r_{tj} r_{t'j} \) times with each of its \( i \) th associates.
if treatments $t$ and $t'$ are $i^{th}$ associates. So,

$$\sum_{j=1}^{b} r_{tj}(k-r_{tj}) = \sum_{i=1}^{u} n_i \lambda_i.$$  

Hence, we obtain:

(3). \[ n_1 \lambda_1 + n_2 \lambda_2 + \ldots + n_u \lambda_u = rk-r^* \]

Also,

(4). \[ p_{j1}^i + p_{j2}^i + \ldots + p_{ju}^i = \begin{cases} n_j - 1 & \text{if } i = j \\ n_j & \text{if } i \neq j \end{cases} \]

(5). \[ n_i p_{jk}^i = n_j p_{ik}^j \]

Now we define a BNAS for an NPBB design having $v = \sum_{i=1}^{n} m_i$ treatments applied in $b$ blocks of $k$ plots of each. In a factorial system of $n$ factors $F_1, F_2, \ldots, F_n$ at $m_1, m_2, \ldots, m_n$ levels respectively, the two treatments are the $p_1 p_2 \ldots p_n^{th}$ associates, where $p_i = 1$, if the $i^{th}$ factor occurs at the same level in both treatments and $p_i = 0$ otherwise; $\lambda_{p_1 p_2 \ldots p_n}$ will denote the number of times these treatments occur together in the same blocks. Suppose two treatments $t$ and $t'$ are $p_1 p_2 \ldots p_n^{th}$ associates and these treatments are replicated $r_{tj}$ and $r_{t'j}$ times in the $j^{th}$ block respectively, then

(5.1) \[ \lambda_{p_1 p_2 \ldots p_n} = \sum_{j=1}^{b} r_{tj} r_{t'j} \]

If $\lambda_{p_1 p_2 \ldots p_n}$ does not depend upon a particular pair of $p_1 p_2 \ldots p_n^{th}$ associates and if $r = \sum_{j=1}^{b} r_{tj}$ is a constant for $t = 1, 2, \ldots, v$, then the above block design is an NPBB design with respect to the "binary number association scheme".
Any contrast belonging to the generalized interaction \( f_{i_1} f_{i_2} \cdots f_{i_n} \) in the above design is estimated with variance

\[(5.2) \quad \sigma^2 / r \theta(q_1, q_2, \ldots, q_n),\]

where \( q_i \) is 0 or 1, then the relation between \( \theta \)'s and \( \lambda \)'s is

\[(5.3) \quad r \prod_{i=1}^{n} \theta_i F_i(1) \cdot \theta = -\frac{1}{k^2}, \]

where

\[(5.4) \quad F_i(1) = \frac{1}{m_i} \begin{pmatrix} 1 & -1 \\ 1 & m_i - 1 \end{pmatrix}, \]

\( \theta = (\theta_0, \theta_1, \ldots, \theta_m)' \), and \( \lambda = (\lambda_0, \lambda_1, \ldots, \lambda_m)' \),

where \( \theta_q \) and \( \lambda_p \) stand for \( \theta(q_1, q_2, \ldots, q_n) \) and \( \lambda_{p_1 p_2 \cdots p_n} \) such that

\[ q = \sum_{h=1}^{n} q_h 2^{n-h} \text{ and } p = \sum_{h=1}^{n} p_h 2^{n-h} \text{ respectively and } \theta_0 = 0 \text{ and } \]

\[ \lambda_m = \sum_{j=1}^{b} t_{j}^2 - rk = r^* - rk. \]

Now we conclude the following:

**Theorem 5.1.** Any NPBB design having BNAS is a FTE and is a PA type block design, and conversely.

**Example 5.1.** Consider the following block design having \( v = 2 \times 3, r = 4, k = 8, \) and \( b = 3 \):
In this design, treatment (00) is a $(1,0)^{th}$ associate with treatments (01) and (02); a $(0,1)^{th}$ associate with treatment (10); and a $(0,0)^{th}$ associate with treatments (11) and (12). Since $r_{(00),1} = 2$, $r_{(00),2} = 1$, $r_{(00),3} = 1$, and $r_{(01),1} = 1$, $r_{(01),2} = 2$, $r_{(01),3} = 1$, then $\lambda_{10} = 5$, and since $r_{(10),1} = 2$, $r_{(10),2} = 1$, $r_{(10),3} = 1$, then $\lambda_{01} = 6$, and lastly, since $r_{(11),1} = 1$, $r_{(11),2} = 2$, $r_{(11),3} = 1$, then $\lambda_{11} = 6$. Also, we obtain $r^* = 6$ and $\lambda_{11} = 6 - 32 = -26$. Hence, the above design is an NPBB design with $v = 2 \times 3$, $r = 4$, $k = 8$, $b = 3$ and $\lambda_{00} = 5$, $\lambda_{01} = 6$, $\lambda_{10} = 5$, $\lambda_{11} = -26$, is an NBFE, and is a PA type block design.

6. **PAB TYPE RECTANGULAR DESIGNS AND n-ARY PARTIALLY BALANCED RECTANGULAR DESIGNS**

Consider a $k \times b$ rectangular design in $n$ factors $F_1, F_2, \ldots, F_n$ at $m_1 m_2 \ldots m_n$ levels having $v = \prod_{i=1}^{n} m_i$ treatments and each treatment being replicated $r$ times. Suppose that the design is a BFE with respect to columns (a PA type column design) and any contrast belonging to the generalized interaction $F_1^{q_1} F_2^{q_2} \ldots F_n^{q_n}$ is estimated with variance

$$\sigma^2/r \delta(q_1, q_2, \ldots, q_n),$$

where $q_i = 0$ or $1$ for $i = 1, 2, \ldots, n$

or

$$\sigma^2/r \delta_q,$$

where $q = \sum_{h=1}^{n} q_h 2^{n-h}$,

then we may obtain the parameter vector $\lambda$ from the following relation:
\[
\mathbf{r} = \frac{1}{k^*} \mathbf{\Phi}_i(1) \cdot \mathbf{a} = \frac{1}{k^*} \mathbf{a},
\]

where \( \mathbf{a} = (0, \theta_2, \ldots, \theta_n)^t \) and \( \mathbf{\lambda} = (\lambda_0, \lambda_1, \ldots, \lambda_{m-1}, r-rk)^t \). Also, suppose that the design is an HBFE with respect to rows, or a PA type row design, and any contrast belonging to generalized interaction \( F_{q_1} F_{q_2} \ldots F_{q_n} \) is estimated with variance

\[
\sigma^2 / r \mathbf{e}_q(q_1, q_2, \ldots, q_n), \text{ where } q_i = 0 \text{ or } 1 \text{ for } i = 1, 2, \ldots, n,
\]

or

\[
\sigma^2 / r \mathbf{e}_q, \text{ where } q = \sum_{h=1}^{n} q_h z^{n-h};
\]

then, we may obtain the parameter vector \( \mathbf{\lambda} \) from the following relation:

\[
\mathbf{r} = \frac{1}{k^*} \mathbf{\Phi}_i(1) \cdot \mathbf{a} = \frac{1}{k^*} \mathbf{a},
\]

where \( \mathbf{a} = (0, \theta_1, \ldots, \theta_n)^t \), \( \mathbf{\lambda} = (\lambda_0, \lambda_1, \ldots, \lambda_{m-1}, r-rk)^t \).

In the above \( k \times b \) rectangular design (from 2.8),

\[
\mathbf{C} = \mathbf{rI}_V - \frac{1}{k} \mathbf{NN}' - \frac{1}{b} \mathbf{NN}' + J \mathbf{V} \mathbf{V}^{-1}
\]

\[
= \left( \mathbf{rI}_V - \frac{1}{k} \mathbf{NN}' \right) + \left( \mathbf{rI}_V - \frac{1}{b} \mathbf{NN}' \right) - \mathbf{rI}_V - \frac{1}{b} \mathbf{VV}^{-1}.
\]

Then, the \( \mathbf{C} \)-matrix can be written as follows:

\[
\mathbf{C} = \sum_{q=1}^{m} r \theta^q \mathbf{L} \mathbf{L}' + \sum_{q=1}^{m} r \mathbf{W} \mathbf{L} \mathbf{L}' - \sum_{q=1}^{m} \mathbf{L} \mathbf{L}'
\]

\[
= \sum_{q=1}^{m} r \left( \theta^q + \mathbf{W}^q - 1 \right) \mathbf{L} \mathbf{L}'.
\]
\[ m = \sum_{q=1}^{n} r_{q}^{*} L_{q}^{*} L_{q}, \]

where \( q^{*} = q + q.1 \) and \( L_{q} \) is a \( v \times n \) \((m_{1}-1)^{q_{1}}\) matrix formed by a complete set of \( n \) normalized orthogonal vectors forming the generalized interaction \( F_{1}F_{2}...F_{n} \). Hence, the above design is a balanced factorial \( k \times b \) rectangular experiment and is a PAB-type \( k \times b \) rectangular design. We now state the following theorem:

**Theorem 6.1.** If the design is an NPBB having BNAS with respect to rows and also to columns, then the design is BFRE and is a PAB-type rectangular design.

From theorem 5.1 and the definition of a PAB-type rectangular design, we obtain:

**Theorem 6.2.** Every PAB-type rectangular design is an NPBB rectangular design having BNAS with respect to rows and to columns, and conversely.

**Remark:** A BFRE is not always a PAB-type rectangular design nor an NPBB design having BNAS with respect to both rows and columns (see Kshirsager [1957]).

**Example 6.1.** Consider the design in Example 5.1 as a \( 3 \times 8 \) rectangular arrangement.

(1). With respect to columns, \( v = 2 \times 3 \), \( r = 4 \), \( k = 3 \), \( b = 8 \), and \( \lambda_{00} = 2 \), \( \lambda_{01} = 0 \), \( \lambda_{10} = 2 \), and \( \lambda_{11} = -8 \).

Using the formula given by Shah [1960],

\[ \theta = -\frac{1}{rk} \sum_{i=1}^{m} G(m_{i}) \cdot \Lambda, \]

where \( G(m_{i}) = \begin{pmatrix} m_{i}-1 \\ -1 \end{pmatrix} \)
Hence,
\[ r(0,1) = 4, \quad r(1,0) = 8/3, \text{ and } r(1,1) = 8/3. \]

(2). With respect to rows, \( v = 2 \times 3, r = 4, k = 8, b = 3, \) and \( \lambda_{00} = 5, \lambda_{01} = 6, \lambda_{10} = 5, \) and \( \lambda_{11} = -26. \)

In this case

\[ \mathbf{\tilde{B}} = -\frac{1}{32} \begin{bmatrix} 2 & 1 & 2 & 1 \\ -1 & 1 & -1 & 1 \\ -2 & -1 & 2 & 1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \\ 5 \\ -26 \end{bmatrix} = \begin{bmatrix} 0 \\ 15/16 \\ 1 \\ 1 \end{bmatrix}. \]

Hence,
\[ r(0,1) = 15/4, \quad r(1,0) = 4, \text{ and } r(1,1) = 4. \]

(3). From (1) and (2), the above design is a "balanced factorial 3x8 rectangular experiment" and is a PAB type rectangular design having the following efficiency factors:

\[ e^*(0,1) = 15/16, \quad e^*(1,0) = 2/3, \text{ and } e^*(1,1) = 2/3. \]
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