RELATION BETWEEN POISSON AND MULTINOMIAL DISTRIBUTIONS

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Introduction. It is usual to see the Poisson distribution developed from the binomial distribution by removing the restriction on the exponent. Here it is shown that the imposition of a fixed total on the number of successes observed in several Poisson populations leads to the multinomial distribution. Fisher (1) has obtained the same result under other circumstances.

Theory. Let \( x_1, \ldots, x_k \) be the numbers of successes observed in \( k \) trials on Poisson populations with means \( \mu_1, \ldots, \mu_k \) respectively. The joint distribution of the observations is

\[
(1) \quad f(x_1, \ldots, x_k) = \frac{e^{-\sum \mu_i} x_1 \mu_1 \cdots x_k \mu_k}{x_1! \cdots x_k!}, \quad x_i = 0, 1, 2, \ldots, i=1, \ldots, k
\]

Set \( \Sigma x_i = n \) and consider (1) as the joint distribution of \( n \) and some \( k-1 \) of the \( x_i \)'s, say \( x_1, \ldots, x_{k-1} \).

For the marginal distribution of \( n \), we require

\[
P(\Sigma x_i = n) = \sum_{x} \frac{e^{-\sum \mu_i} x_1 \mu_1 \cdots x_k \mu_k}{x_1! \cdots x_k!}
\]

where \( \sum \) is the sum for all configurations of the \( x_i \) such that \( \Sigma x_i = n \).

From

\[
(\mu_1 + \cdots + \mu_k)^n = \sum \frac{n!}{x_1! \cdots x_k!} \mu_1^{x_1} \cdots \mu_k^{x_k}
\]

where \( \Sigma x_i = n \), we have

\[
\frac{(\mu_1 + \cdots + \mu_k)^n}{n!} = \sum \frac{\mu_1^{x_1} \cdots \mu_k^{x_k}}{x_1! \cdots x_k!}
\]

Hence

\[
P(\Sigma x_i = n) = \frac{e^{-\sum \mu_i} (\Sigma \mu_i)^n}{n!}
\]
and the marginal distribution of $n$ is

$$f(n) = \frac{e^{-\sum_{i} \mu_i} (\sum_{i} \mu_i)^n}{n!}, \quad n = 0, 1, 2, \ldots,$$

a Poisson distribution with parameter $\sum_{i} \mu_i$.

From (1) and (2), the conditional distribution of $x_1, \ldots, x_{k-1}$ is

$$f(x_1, \ldots, x_{k-1}|n) = \frac{n!}{x_1! \cdots x_{k-1}!} \frac{\mu_i}{(\sum_{i} \mu_i)}^{x_k}, \quad \text{with } x_k = n - \sum_{i=1}^{k-1} x_i.$$

This is clearly a multinomial distribution.

REFERENCE