# Minimax estimation of the reciprocal of population size in a two-sample mark-recapture experiment: Preliminary Report

D. S. Robson

BU-377-M June, 1971

## Abstract

A constant risk minimax estimator of the reciprocal of population size is conjectured to exist for the case of a two-sample mark-recapture experiment when loss is measured by squared error. For samples of size $n = 1$ from a population containing D marked members this estimator exists and takes the value $1/(12D)$ when the sampled item is unmarked, and takes the value $5/(12D)$ when the samples item is a recapture; the constant risk is $1/(12D)^2$. For $n = 2$ the constant risk minimax estimator is obtained from the solution of a cubic equation.

Minimax estimation of the reciprocal of population size

in a two-sample mark-recapture experiment: Preliminary Report

D. S. Robson

BU-377-M

June, 1971

## INTRODUCTION

One of the earliest solutions to a minimax point estimation problem concerned estimation of the proportion of "defective" $\left(\frac{D}{N}\right)$ in a lot of N items from which n are randomly selected and X are observed to be defective. In this case a linear function aX + b can be constructed which has constant mean squared error for all integer values D, $0 \le D \le N$. The two-sample mark-recapture problem is dual of the above problem in the sense that N rather than D is the unknown parameter, and since survival rate is given by

$$S = \frac{1}{N} \text{ (number of survivors at some later date)}$$

there is some value constructing a minimax estimator of 1/N or, since D is a known constant, a minimax estimator of D/N.

We offer the conjecture that a minimax estimator also exists for this situation, having constant mean squared error for all integer values of $N \ge \max(D,n)$. In support of the conjecture we construct such an estimator for n = 1 and n = 2.

## MINIMAX ESTIMATOR OF 1/N WHEN n = 1 AND n = 2

The probability distribution of X when n = 1 is given by $p_0 = (N-D)/N$ and $p_1 = D/N$, so the mean squared error of an estimator $T_X$ of 1/N is

$$E\left(T_X-\frac{1}{N}\right)^2 = \frac{N-D}{N}\left(T_0-\frac{1}{N}\right)^2 + \frac{D}{N}\left(T_1-\frac{1}{N}\right)^2 \quad .$$

If this risk function is to be constant with respect to N,

$$E\left(T_X-\frac{1}{N}\right)^2 \equiv C^2$$

then

$$(N-D)\left(N^2T_0^2-2NT_0+1\right) + D\left(N^2T_1^2-2NT_1+1\right) \underset{N}{\equiv} N^3C^2$$

or

$$N^3\left(T_0^2-C^2\right) + N^2\left(T_1^2D-T_0^2D-2T_0\right) + N\left(1-2T_0D-2T_1D\right) \underset{N}{\equiv} 0 \quad .$$

Thus, the coefficients of $N^3, N^2$ and N in this polynomial must all be zero, giving

$$T_0 = C = \frac{1}{12D} \qquad\qquad T_1 = \frac{5}{12D} \quad .$$

For n = 2 we obtain the corresponding polynomial in N with coefficients depending on $T_0$, $T_1$, $T_2$ and C:

$$N^4\left(T_0^2-C^2\right) + N^3\left[2T_1^2D - T_0^2(2D+1) - 2T_0 + C^2\right]$$

$$+ N^2\left[T_2^2D(D-1) - 2T_1^2D^2 - 4T_1D + T_0^2D(D+1) + 2T_0(2D+1) + 1\right]$$

$$- N\left[2T_2D(D-1) - 4T_1D^2 + 2T_0D(D+1) + 1\right] \underset{N}{\equiv} 0 \quad .$$

In terms of C the solution is

$$T_0 = C \qquad T_1 = \sqrt{C^2 + C/D} \qquad T_2 = \frac{\left[4D\sqrt{C^2D^2 + CD} - 2CD(D+1) - 1\right]}{2D(D-1)}$$

and C is determined as the root of a cubic equation.