Time to Extinction of a Pure Death Process

by

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Abstract

The expected time to extinction of a homogeneous pure death process is calculated in two ways. The fact that the two results must be equal establishes a proof of the identity $\sum_{i=1}^{n} \frac{1}{i} = \sum_{i=1}^{n} (-1)^{i-1} \binom{n}{i} \frac{1}{i}$.

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Suppose that $\{N(t);\ t \geq 0\}$ is a homogeneous pure death process with constant force of mortality μ and initial size $N(0) = N_o$. Then for $n = 0,1,2,\ldots,N_o$ and $t \geq 0$,

$$P[N(t) = n] = \binom{N_o}{n} e^{-\mu nt}\left(1-e^{-\mu t}\right)^{N_o-n} .$$

Let T be the time of extinction of this process. Then the c.d.f. of T is

$$G_T(t) = P[T \leq t] = P[N(t) = 0] = \left(1-e^{-\mu t}\right) \quad , \ t \geq 0$$

with corresponding p.d.f.

$$g_T(t) = \frac{d}{dt}G_T(t) = N_o \mu e^{-\mu t}\left(1-e^{-\mu t}\right)^{N_o-1} \quad , \ t \geq 0 .$$

Now $ET = N_o\mu \int_0^\infty t e^{-\mu t}\left(1-e^{-\mu t}\right)^{N_o-1} dt$

$$= N_o\mu \sum_{i=0}^{N_o-1} \binom{N_o-1}{i}(-1)^i \int_0^\infty t e^{-\mu t(i+1)} dt$$

$$= N_o\mu \sum_{i=0}^{N_o-1} \binom{N_o-1}{i}(-1)^i \frac{1}{\mu^2(i+1)^2} = \frac{N_o}{\mu} \sum_{i=0}^{N_o-1} \frac{\binom{N_o-1}{i}}{(i+1)^2}(-1)^i$$

$$= \frac{1}{\mu}\sum_{i=1}^{N_o} (-1)^i \binom{N_o}{i} \frac{1}{i} .$$

Alternatively, let T_i be the time to <u>first</u> death in a population of initial size i undergoing a pure death process with parameter μ. Then T_i has c.d.f.

$$F_i(t) \equiv P\left[T_i \le t\right] = 1-P\left[T_i > t\right] = 1-P\left[N(t) = i \mid N(o) = i\right]$$

$$= 1-e^{-i\mu t} \quad , \ t \ge 0 \ ,$$

with corresponding density

$$f_i(t) = i\mu e^{-i\mu t} \quad , \ t \ge 0 \ \text{ and}$$

$$ET_i = \frac{1}{i\mu} \quad .$$

Now the time to extinction, T, of a population of initial size N_o, is the sum of times between the N_o successive deaths. i.e., $T = T_{N_o} + T_{N_o-1} + \ldots + T_1$. Thus $ET = \sum_{i=1}^{N_o} ET_i = \frac{1}{\mu} \sum_{i=1}^{N_o} \frac{1}{i}$.

The fact that these two results must be equal establishes the identity

$$\sum_{i=1}^{n} \frac{1}{i} = \sum_{i=1}^{n} (-1)^i \binom{n}{i} \frac{1}{i}$$

for every positive integer n . For other proofs of this identity see Feller (1950), p. 63.

REFERENCE

Feller, William (1950). <u>An Introduction to Probability Theory and Its Applications</u>. 2nd ed. John Wiley and Sons, New York.