ATTEMPTS AT INVERTING THE VARIANCE-COVARIANCE MATRIX OF THE 2-WAY CROSSED CLASSIFICATION, UNBALANCED DATA, RANDOM MODEL.

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J. W. Rudan and S. R. Searle

Abstract

Elements of the information matrix for the variance components of a linear model involve, under normality, the inverse of the variance-covariance matrix of the vector of observations. Attempts at finding this inverse for the 2-way crossed classification random model, unbalanced data, are described.

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1. The Model

The equation of the model of the 2-way crossed classification with unbalanced data is taken as

$$y_{ijk} = \mu + \alpha_i + \beta_j + \gamma_{ij} + e_{ijk} \tag{1}$$

with $i = 1, 2, \cdots, a$, $j = 1, 2, \cdots, b$, and $k = 1, 2, \cdots, n_{ij}$, where α_i and β_j are the i^{th} and j^{th} effects respectively of the α- and β-classifications, with γ_{ij} being the corresponding interaction effect, and e_{ijk} is the random error term. For the random model all elements of (1), except the mean μ, are assumed to be random, with zero means and variances σ^2_α, σ^2_β, σ^2_γ, and σ^2_e respectively. All covariances between unlike elements are assumed zero. For convenience we write

$$\begin{matrix} \alpha \equiv \sigma^2_\alpha & \gamma \equiv \sigma^2_r \\ \beta \equiv \sigma^2_\beta & e \equiv \sigma^2_e \end{matrix} , \tag{2}$$

and also let

$\underline{y}$ = vector of observations y_{ijk} listed in lexicon order

with

$\underline{V} = \text{var}(\underline{y})$.

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Then the elements of $\underline{V}$ are zero and various sums of the variances (2).

Example

Suppose that for a = 2 and b = 3 the values of the n_{ij}'s are as follows:

n_{ij}

	j = 1	j = 2	j = 3	$n_{i.}$
i = 1	1	3	2	6
i = 2	2	4	2	8
$n_{.j}$	3	7	4	14 = $n_{..}$

Then

$$\underline{y}' = (y_{111}\ y_{121}\ y_{122}\ y_{123}\ y_{131}\ y_{132}\ y_{211}\ y_{212}\ y_{221}\ y_{222}\ y_{223}\ y_{224}\ y_{231}\ y_{232}),$$

i.e. the elements in $\underline{y}$ are the y_{ijk} ordered by k within j within i (lexicon order). On writing

$$v = \alpha + \beta + \gamma + e \equiv \sigma^2_\alpha + \sigma^2_\beta + \sigma^2_\gamma + \sigma^2_e$$

and

$$c = \alpha + \beta + \gamma \equiv \sigma^2_\alpha + \sigma^2_\beta + \sigma^2_\gamma \tag{3}$$

the 14 × 14 matrix $\underline{V}$ is

$$\underline{V} = \left[\begin{array}{cccccc|cccccccc}
v & \alpha & \alpha & \alpha & \alpha & \alpha & \beta & \beta & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\alpha & v & c & c & \alpha & \alpha & \cdot & \cdot & \beta & \beta & \beta & \beta & \cdot & \cdot \\
\alpha & c & v & c & \alpha & \alpha & \cdot & \cdot & \beta & \beta & \beta & \beta & \cdot & \cdot \\
\alpha & c & c & v & \alpha & \alpha & \cdot & \cdot & \beta & \beta & \beta & \beta & \cdot & \cdot \\
\alpha & \alpha & \alpha & \alpha & v & c & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \beta & \beta \\
\alpha & \alpha & \alpha & \alpha & c & v & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \beta & \beta \\
\hline
\beta & \cdot & \cdot & \cdot & \cdot & \cdot & v & c & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha \\
\beta & \cdot & \cdot & \cdot & \cdot & \cdot & c & v & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha \\
\cdot & \beta & \beta & \beta & \cdot & \cdot & \alpha & \alpha & v & c & c & c & \alpha & \alpha \\
\cdot & \beta & \beta & \beta & \cdot & \cdot & \alpha & \alpha & c & v & c & c & \alpha & \alpha \\
\cdot & \beta & \beta & \beta & \cdot & \cdot & \alpha & \alpha & c & c & v & c & \alpha & \alpha \\
\cdot & \beta & \beta & \beta & \cdot & \cdot & \alpha & \alpha & c & c & c & v & \alpha & \alpha \\
\cdot & \cdot & \cdot & \cdot & \beta & \beta & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & v & c \\
\cdot & \cdot & \cdot & \cdot & \beta & \beta & \alpha & \alpha & \alpha & \alpha & \alpha & \alpha & c & v
\end{array}\right] \tag{4}$$

where a dot represents zero. Partitioned into a^2 sub-matrices, those on the diagonal having order $n_{i.} \times n_{i.}$ correspond to the $n_{i.}$ observations in the i^{th} level of the α-factor. Accordingly, these matrices have all elements equal to $\alpha = \sigma^2_\alpha$, save for diagonal sub-matrices $v\underline{I} + c\underline{J} - c\underline{I}$ of order $n_{ij} \times n_{ij}$ corresponding to the n_{ij} observations in the ij-cell. The rectangular matrices of β's have order $n_{ij} \times n_{i'j}$ for $i \neq i'$ and lie "corner to corner" as shown in (4).

Note that in this example all $n_{ij} > 0$. If some $n_{ij} = 0$, corresponding matrices of β's do not lie "corner to corner". For example, had n_{22} been zero the upper right-hand sub-matrix of (4) would have been

$$\begin{bmatrix} \beta & \beta & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \beta & \beta \\ \cdot & \cdot & \beta & \beta \end{bmatrix} .$$

Absence of the "corner to corner" property evident in (4) complicates the discussion that follows. We therefore confine ourselves to cases where all $n_{ij} > 0$. [If explicit results can be obtained for this case we should be able to put $n_{ij} = 0$ where needed and obtain results for that case.]

Extension of $\underline{V}$ from (4) to the general case is apparent, and leads to the following definition of $\underline{V}$:

$$\underline{V} = \left\{\underline{V}_{ij,i'j'}\right\} \text{ for } i,i' = 1, 2, \cdots, a, \text{ and } j,j' = 1, 2, \cdots, b,$$

where

$$\begin{aligned}
\underline{V}_{ij,ij} &= (v - c)\underline{I}_{n_{ij}} + c\underline{J}_{n_{ij}\times n_{ij}} = e\underline{I}_{n_{ij}} + c\underline{J}_{n_{ij}\times n_{ij}} \\
\underline{V}_{ij,ij'} &= \alpha\underline{J}_{n_{ij}\times n_{ij'}} && \text{for } j' \neq j \\
\underline{V}_{ij,i'j} &= \beta\underline{J}_{n_{ij}\times n_{i'j}} && \text{for } i' \neq i \qquad (5) \\
\underline{V}_{ij,i'j'} &= 0\underline{J}_{n_{ij}\times n_{i'j'}} && \text{for } i' \neq i \text{ and } j' \neq j
\end{aligned}$$

and the $\underline{J}$-matrices have all elements unity.

2. Maximum Likelihood

Analysis of variance estimators of the variance components (2) and the sampling variances of these estimators under normality assumptions are discussed in Searle (1958). Although maximum likelihood estimators cannot be obtained explicitly, Hartley and Rao (1967) have an iterative procedure for deriving them numerically. Also, the large sample sampling variances of the maximum likelihood estimators are, from Searle (1970), the elements of the 4×4 matrix

$$2\underline{T}^{-1} = 2\left\{\mathrm{tr}\left(\underline{V}^{-1}\frac{\partial \underline{V}}{\partial \sigma_i^2}\underline{V}^{-1}\frac{\partial \underline{V}}{\partial \sigma_j^2}\right)\right\}^{-1} \tag{6}$$

$$\text{for } \sigma_i^2,\ \sigma_j^2 = \sigma_\alpha^2,\ \sigma_\beta^2,\ \sigma_\gamma^2,\ \sigma_e^2 \equiv \alpha,\ \beta,\ \gamma,\ e.$$

From (5), the derivatives of $\underline{V}$ w.r.t. the σ^2's are easily obtained:

$$\frac{\partial \underline{V}}{\partial \alpha} = \sum_{i=1}^{a}{}^{+}\ \underline{J}_{n_{i\cdot} \times n_{i\cdot}}$$

$$\frac{\partial \underline{V}}{\partial \beta} = \left\{\delta_{jj'}\underline{J}_{n_{ij} \times n_{i'j'}}\right\} \quad \text{for } i,i' = 1, 2, \cdots, a, \text{ and } j,j' = 1, 2, \cdots, b;$$

$$\frac{\partial \underline{V}}{\partial \gamma} = \sum_{i=1}^{a}{}^{+}\sum_{j=1}^{b}{}^{+}\ \underline{J}_{n_{ij} \times n_{ij}} \tag{7}$$

$$\frac{\partial \underline{V}}{\partial e} = \underline{I}_{n_{\cdot\cdot}}\ ,$$

where Σ^+ represents the operation of direct sum and $\delta_{jj'}$ is the Kronecker delta, $\delta_{jj} = 1$ and $\delta_{jj'} = 0$ for $j \neq j'$. Demonstration of these results can be gotten from the example of $\underset{\sim}{V}$ given in (4).

The difficult part of (6) is obtaining $\underset{\sim}{V}^{-1}$. To this we now turn.

3. Inverting V

The following lemma taken from Urquhart (1962) and quoted in Searle (1970) is used.

Lemma: For the matrix $\underset{\sim}{A}$ partitioned as

$$\underset{\sim}{A} = \left\{\underset{\sim}{A}_{pq} \text{ of order } n_p \times n_q\right\} \text{ for } p,q = 1, 2, \cdots, N \tag{8}$$

where

$$\underset{\sim}{A}_{pq} = \delta_{pq} b_p \underset{\sim}{I}_{n_p \times n_q} + g_{pq} \underset{\sim}{J}_{n_p \times n_q} \tag{9}$$

with

$$\underset{\sim}{G} = \left\{g_{pq}\right\} , \tag{10}$$

then

$$\underset{\sim}{A}^{-1} = \left\{(\underset{\sim}{A}^{-1})_{pq} \text{ of order } n_p \times n_q\right\} \tag{11}$$

with

$$(\underset{\sim}{A}^{-1})_{pq} = \delta_{pq}(1/b_p)\underset{\sim}{I}_{n_p \times n_q} + h_{pq}\underset{\sim}{J}_{n_p \times n_q} \tag{12}$$

where

$$\underset{\sim}{H} = \left\{h_{pq}\right\} = \left[(\underset{\sim}{G}\underset{\sim}{D} + \underset{\sim}{B})^{-1} - \underset{\sim}{B}^{-1}\right]\underset{\sim}{D}^{-1} \tag{13}$$

with

$$\underset{\sim}{D} = \text{diag}\{n_p\} = \text{diag}\{n_1\ n_2\ \cdots\ n_N\} \tag{14}$$

and

$$\underset{\sim}{B} = \text{diag}\{b_p\} = \text{diag}\{b_1\ b_2\ \cdots\ b_N\}. \tag{15}$$

Note that in (13)

$$\underset{\sim}{H} = \underset{\sim}{D}^{-1}(\underset{\sim}{G} + \underset{\sim}{B}\underset{\sim}{D}^{-1})^{-1}\underset{\sim}{D}^{-1} - \underset{\sim}{B}^{-1}\underset{\sim}{D}^{-1} . \tag{16}$$

To use the above lemma for inverting $\underset{\sim}{V}$ of (5) the terms of the lemma are, by associating (5) with (8), (9), (10), (14), and (15),

$$p,q = (ij),\ (i'j') \quad \text{for } i,i' = 1, 2, \cdots, a, \text{ and } j,j' = 1, 2, \cdots, b;$$

$$b_p = e \quad \text{for all } p,$$

$$\underline{G} = \left\{\beta\underline{I}_b + \delta_{ii'}(\gamma\underline{I}_b + \alpha\underline{J}_b)\right\} \quad \text{for } i,i' = 1, 2, \cdots, a; \tag{17}$$

$$\underline{B} = e\underline{I}_{ab}$$

and

$$\underline{D} = \text{diag}\left\{n_{11}\ n_{12} \cdots n_{1b}\ n_{21} \cdots n_{2b} \cdots \quad \cdots n_{a1} \cdots n_{ab}\right\} . \tag{18}$$

The example in (4) illustrates these associations.

From (11), (12), and (16) we then get $\underline{V}^{-1}$ as follows:

$$\underline{V}^{-1} = \left\{(\underline{V}^{-1})_{ij,i'j'} \text{ of order } n_{ij} \times n_{i'j'}\right\} \tag{18a}$$

with

$$(\underline{V}^{-1})_{ij,i'j'} = \delta_{ij,i'j'}(1/e)\underline{I}_{n_{ij} \times n_{i'j'}} + h_{ij,i'j'}\underline{J}_{n_{ij} \times n_{i'j'}} \tag{18b}$$

where

$$\underline{H} = \underline{D}^{-1}(\underline{G} + e\underline{D}^{-1})^{-1}\underline{D}^{-1} - (1/e)\underline{D}^{-1} \tag{19}$$

using (17) and (18) for $\underline{G}$ and $\underline{D}$, respectively. In (19) the difficult term is $(\underline{G} + e\underline{D}^{-1})^{-1}$ which we write as $\underline{F}^{-1}$; i.e.,

$$\underline{F} = \underline{G} + e\underline{D}^{-1} \tag{20}$$

$$= \left\{\underline{F}_{ii'} \text{ of order } b \times b\right\} \quad \text{for } i,i' = 1, 2, \cdots, a .$$

Then from (17) and (18)

$$\underline{F}_{ii} = \beta\underline{I}_b + \gamma\underline{I}_b + \alpha\underline{J}_b + e\ \text{diag}\left\{\frac{1}{n_{i1}} \cdots \frac{1}{n_{ib}}\right\}$$

$$= \text{diag}\left\{\beta + \gamma + e/n_{ij}\right\} \quad \text{for } j = 1, 2, \cdots, b + \alpha\underline{J}_b$$

$$= \beta\left[\text{diag}\left\{1 + (n_{ij}\gamma + e)/n_{ij}\beta\right\} \quad \text{for } j = 1,2,\cdots,b + \alpha/\beta\underline{J}_b\right] \tag{21}$$

which may be written variously as

$$\underset{\sim}{F}_{ii} = \beta(\underset{\sim}{I} + \underset{\sim}{D}_i + \lambda\underset{\sim}{J}) \tag{22}$$

$$= \beta(\underset{\sim}{I} + \underset{\sim}{K}_i) \tag{23}$$

$$= \beta\underset{\sim}{L}_i \tag{24}$$

where

$$\underset{\sim}{D}_i = \text{diag}\{d_{ij}\} \quad \text{for } j = 1, 2, \cdots, b, \text{ with } d_{ij} = (n_{ij}\gamma + e)/n_{ij}\beta; \tag{25}$$

$$\lambda = \alpha/\beta, \tag{26}$$

$$\underset{\sim}{K}_i = \underset{\sim}{D}_i + \lambda\underset{\sim}{J} \tag{27}$$

and

$$\underset{\sim}{L}_i = \underset{\sim}{I} + \underset{\sim}{K}_i \;. \tag{28}$$

Further, from (17) and (20)

$$\underset{\sim}{F}_{ii'} = \beta\underset{\sim}{I} \quad \text{for } i \neq i' \;. \tag{29}$$

From (23) and (29), the matrix $\underset{\sim}{F}$ of (20) is therefore

$$\underset{\sim}{F} = \beta\begin{bmatrix} \underset{\sim}{I} + \underset{\sim}{K}_1 & \underset{\sim}{I} & \underset{\sim}{I} & \cdots & \underset{\sim}{I} \\ \underset{\sim}{I} & \underset{\sim}{I} + \underset{\sim}{K}_2 & \underset{\sim}{I} & \cdots & \underset{\sim}{I} \\ \underset{\sim}{I} & \underset{\sim}{I} & \underset{\sim}{I} + \underset{\sim}{K}_3 & \cdots & \underset{\sim}{I} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \underset{\sim}{I} & \underset{\sim}{I} & \underset{\sim}{I} & \cdots & \underset{\sim}{I} + \underset{\sim}{K}_a \end{bmatrix} \tag{30}$$

with $\underset{\sim}{K}_i$ being defined by (27) and (25); or, using $\underset{\sim}{L}_i = \underset{\sim}{I} + \underset{\sim}{K}_i$, we have $\underset{\sim}{F}$ as

$$\underset{\sim}{F} = \beta\begin{bmatrix} \underset{\sim}{L}_1 & \underset{\sim}{I} & \underset{\sim}{I} & \cdots & \underset{\sim}{I} \\ \underset{\sim}{I} & \underset{\sim}{L}_2 & \underset{\sim}{I} & \cdots & \underset{\sim}{I} \\ \underset{\sim}{I} & \underset{\sim}{I} & \underset{\sim}{L}_3 & \cdots & \underset{\sim}{I} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \underset{\sim}{I} & \underset{\sim}{I} & \underset{\sim}{I} & \cdots & \underset{\sim}{L}_a \end{bmatrix} \tag{31}$$

Note that in (30) and (31) the sub-matrices are all square of order b.

The problem now is to invert $\underset{\sim}{F}$. By the definition of $\underset{\sim}{F}$ in (20), $\underset{\sim}{F}^{-1}$, once found, is used in (19) to give

$$\underset{\sim}{H} = \underset{\sim}{D}^{-1}\underset{\sim}{F}^{-1}\underset{\sim}{D}^{-1} - (1/e)\underset{\sim}{D}^{-1} \tag{31a}$$

where

$$\underset{\sim}{D}^{-1} = \mathrm{diag}\{1/n_{ij}\} \quad \text{from (18)}$$

and $\underset{\sim}{H}$ is used in (18b) to give $\underset{\sim}{V}^{-1}$ in (18a).

Amendment of a suggestion made by Thompson (1970) gives the inverse of $\underset{\sim}{F}$ of (31) as

$$\underset{\sim}{F}^{-1} = (1/\beta)\{\underset{\sim}{E}_{ii'}\} \quad \text{for } i,i' = 1, 2, \cdots, a \tag{32}$$

with

$$\underset{\sim}{E}_{ii} = (\underset{\sim}{L}_i - \underset{\sim}{I})^{-1} - (\underset{\sim}{L}_i - \underset{\sim}{I})^{-1}\underset{\sim}{Q}^{-1}(\underset{\sim}{L}_i - \underset{\sim}{I})^{-1}$$

and

$$\underset{\sim}{E}_{ii'} = -(\underset{\sim}{L}_i - \underset{\sim}{I})^{-1}\underset{\sim}{Q}^{-1}(\underset{\sim}{L}_{i'} - \underset{\sim}{I})^{-1} \quad \text{for } i \neq i'$$

where

$$\underset{\sim}{Q} = \underset{\sim}{I} + \sum_{i=1}^{a} (\underset{\sim}{L}_i - \underset{\sim}{I})^{-1} .$$

In view of (28) this becomes

$$\underset{\sim}{E}_{ii} = \underset{\sim}{K}_i^{-1} - \underset{\sim}{K}_i^{-1}\underset{\sim}{Q}^{-1}\underset{\sim}{K}_i^{-1} \tag{33}$$

and

$$\underset{\sim}{E}_{ii'} = -\underset{\sim}{K}_i^{-1}\underset{\sim}{Q}^{-1}\underset{\sim}{K}_{i'}^{-1} \quad \text{for } i \neq i' \tag{34}$$

with

$$\underset{\sim}{Q} = \underset{\sim}{I} + \sum_{i=1}^{a} \underset{\sim}{K}_i^{-1} . \tag{35}$$

Proof of inverse: Ignoring the factor β in $\underline{F}$ of (30) and the $1/\beta$ in (32), we have

$$\sum_{i'=1}^{a} \underline{F}_{ii'}\underline{E}_{i'i} = (\underline{I} + \underline{K}_i)(\underline{K}_i^{-1} - \underline{K}_i^{-1}\underline{Q}^{-1}\underline{K}_i^{-1}) - \sum_{i'\neq i}^{a} \underline{K}_{i'}^{-1}\underline{Q}^{-1}\underline{K}_i^{-1}$$

$$= \underline{K}_i^{-1} + \underline{I} - \underline{K}_i^{-1}\underline{Q}^{-1}\underline{K}_i^{-1} - \underline{Q}^{-1}\underline{K}_i^{-1} - \sum_{i'\neq i}^{a} \underline{K}_{i'}^{-1}\underline{Q}^{-1}\underline{K}_i^{-1}$$

$$= \underline{I} + \underline{K}_i^{-1} - \underline{Q}^{-1}\underline{K}_i^{-1} - \sum_{i'=1}^{a} \underline{K}_{i'}^{-1}\underline{Q}^{-1}\underline{K}_i^{-1}$$

$$= \underline{I} + (\underline{I} - \underline{Q}^{-1})\underline{K}_i^{-1} - (\underline{Q} - \underline{I})\underline{Q}^{-1}\underline{K}_i^{-1}$$

$$= \underline{I}$$

and, for $i < i''$, which implies no lack of generality,

$$\sum_{i'=1}^{a} \underline{F}_{ii'}\underline{E}_{i'i''} = - \sum_{i'=1}^{i-1} \underline{K}_{i'}^{-1}\underline{Q}^{-1}\underline{K}_{i''}^{-1} + (\underline{I} + \underline{K}_i)(- \underline{K}_i^{-1}\underline{Q}^{-1}\underline{K}_{i''}^{-1})$$

$$- \sum_{i'=i+1}^{i''-1} \underline{K}_{i'}^{-1}\underline{Q}^{-1}\underline{K}_{i''}^{-1} + \underline{K}_{i''}^{-1} - \underline{K}_{i''}^{-1}\underline{Q}^{-1}\underline{K}_{i''}^{-1}$$

$$- \sum_{i'=i''+1}^{a} \underline{K}_{i'}^{-1}\underline{Q}^{-1}\underline{K}_{i''}^{-1}$$

$$= - \sum_{i'=1}^{a} \underline{K}_{i'}^{-1}\underline{Q}^{-1}\underline{K}_{i''}^{-1} + \underline{K}_i^{-1}\underline{Q}^{-1}\underline{K}_{i''}^{-1} + \underline{K}_{i''}^{-1}\underline{Q}^{-1}\underline{K}_{i''}^{-1}$$

$$- \underline{K}_i^{-1}\underline{Q}^{-1}\underline{K}_{i''}^{-1} - \underline{Q}^{-1}\underline{K}_{i''}^{-1} + \underline{K}_{i''}^{-1} - \underline{K}_{i''}^{-1}\underline{Q}^{-1}\underline{K}_{i''}^{-1}$$

$$= - (\underline{Q} - \underline{I})\underline{Q}^{-1}\underline{K}_{i''}^{-1} + (\underline{I} - \underline{Q}^{-1})\underline{K}_{i''}^{-1}$$

$$= \underline{0} \; .$$

Equations (32) - (35) thus give $\underline{F}^{-1}$. They involve $\underline{K}_i^{-1}$ which, from (27), is

$$\underline{K}_i^{-1} = (\underline{D}_i + \lambda\underline{J})^{-1} . \tag{36}$$

This inverse is a special case of the result

$$(\underline{A} + \lambda\underline{x}\underline{y}')^{-1} = \underline{A}^{-1} - \frac{\lambda}{1 + \lambda\underline{y}'\underline{A}^{-1}\underline{x}} (\underline{A}^{-1}\underline{x}\underline{y}'\underline{A}^{-1}) \tag{37}$$

which can be verified by multiplication as follows, making use of the fact that $\underline{x}\underline{y}'\underline{A}^{-1}\underline{x}\underline{y}'\underline{A}^{-1} = \underline{y}'\underline{A}^{-1}\underline{x}(\underline{x}\underline{y}'\underline{A}^{-1})$ because $\underline{y}'\underline{A}^{-1}\underline{x}$ is a scalar:

$$(\underline{A} + \lambda\underline{x}\underline{y}')\left(\underline{A}^{-1} - \frac{\lambda\underline{A}^{-1}\underline{x}\underline{y}'\underline{A}^{-1}}{1 + \lambda\underline{y}'\underline{A}^{-1}\underline{x}}\right)$$

$$= \underline{I} + \lambda\underline{x}\underline{y}'\underline{A}^{-1} - \frac{\lambda}{1 + \lambda\underline{y}'\underline{A}^{-1}\underline{x}} (\underline{x}\underline{y}'\underline{A}^{-1} + \lambda\underline{x}\underline{y}'\underline{A}^{-1}\underline{x}\underline{y}'\underline{A}^{-1})$$

$$= \underline{I} + \lambda\underline{x}\underline{y}'\underline{A}^{-1} - \frac{\lambda}{1 + \lambda\underline{y}'\underline{A}^{-1}\underline{x}} \left[\underline{x}\underline{y}'\underline{A}^{-1} + \lambda(\underline{y}'\underline{A}^{-1}\underline{x})\underline{x}\underline{y}'\underline{A}^{-1}\right]$$

$$= \underline{I} .$$

To apply (37) to (36) write $\underline{J} = \underline{1}\underline{1}'$, and then

$$\underline{K}_i^{-1} = \underline{D}_i^{-1} - \frac{\lambda}{1 + \lambda\underline{1}'\underline{D}_i^{-1}\underline{1}} (\underline{D}_i^{-1}\underline{1})(\underline{D}_i^{-1}\underline{1})' \tag{38}$$

$$= \text{diag}\{1/d_{i1} \cdots 1/d_{ib}\} - \frac{\lambda}{1 + \lambda \sum_{j=1}^{b} 1/d_{ij}} \left\{\frac{1}{d_{ij}d_{ij'}}\right\} \text{ for } j,j' = 1,2,\cdots,b. \tag{39}$$

Denote the vector of diagonal elements in $\underline{D}_i^{-1}$ by $\underline{r}_i$:

$$\underline{r}_i' = (1/d_{i1} \cdots 1/d_{ib}).$$

Then

$$\underline{K}_i^{-1} = \underline{D}_i^{-1} - \frac{\lambda}{1 + \lambda\underline{r}_i'\underline{1}} \underline{r}_i\underline{r}_i' . \tag{40}$$

In these results we have, as in (25) and (26),

$$d_{ij} = (n_{ij}\gamma + e)/n_{ij}\beta \quad \text{and} \quad \lambda = \alpha/\beta \ .$$

Whilst for $\underline{F}^{-1}$ of (32) - (35) we now have $\underline{K}_i^{-1}$, we also need the inverse of

$$\underline{Q} = \underline{I} + \sum_{i=1}^{a} \underline{K}_i^{-1}$$

$$= \underline{I} + \sum_{i=1}^{a} \underline{D}_i^{-1} - \lambda \sum_{i=1}^{a} \frac{1}{1 + \lambda \underline{r}_i'\underline{1}} \underline{r}_i\underline{r}_i'$$

$$= \text{diag}\left\{1 + \sum_{i=1}^{a} 1/d_{i1} \cdots 1 + \sum_{i=1}^{a} 1/d_{ib}\right\} - \lambda \sum_{i=1}^{a} \frac{1}{1 + \lambda \sum_{j=1}^{b} 1/d_{ij}} \underline{r}_i\underline{r}_i' \ . \qquad (41)$$

Attempts at inverting this have so far failed.

4. Validation

For partial validation we consider the simplest case, namely balanced data with no interaction: $n_{ij} = 1$ for all i and j and $\gamma = 0$. Although the ultimate results, the sampling variances of variance component estimators, are known in this case and do not need to be derived again we will verify that $\underline{V}^{-1}$ of the above procedure is the same as that given by Wallace and Hussain (1969).

Putting $n_{ij} = 1$ and $\gamma = 0$ in (17) and (18) gives

$$\underline{G} \ = \beta\underline{I}_b + \delta_{ii'}\alpha\underline{J}_b \quad \text{for } i,i' = 1, 2, \cdots, a,$$

and

$$\underline{D} \ = \underline{I}_b$$

and in (18b)

$$\underline{V}^{-1} = (1/e)\underline{I}_b + \underline{H} \ .$$

But in (19) and (20)

$$\underline{H} = \underline{F}^{-1} - (1/e)\underline{I}_b$$

so that

$$\underline{V}^{-1} = \underline{F}^{-1} \tag{42}$$

which, in (32) is

$$\underline{V}^{-1} = \underline{F}^{-1} = (1/\beta)\{\underline{E}_{ii'}\} \quad \text{for } i,i' = 1, 2, \cdots, a.$$

From (25) and (27)

$$\underline{K}_i = \underline{K} = \frac{e}{\beta}\underline{I} + \frac{\alpha}{\beta}\underline{J}$$

so that for use in (33) - (35)

$$\underline{K}_i^{-1} = \underline{K}^{-1} = \frac{\beta}{e}\left(\underline{I} - \frac{\alpha}{b\alpha + e}\underline{J}\right),$$

this result coming from the general result

$$(x\underline{I}_n + y\underline{J}_n)^{-1} = (1/x)\underline{I}_n - \frac{y}{x(x + ny)}\underline{J}_n$$

given, for example, in Searle (1966, p. 198). Then, in (35)

$$\underline{Q} = \underline{I} + a\underline{K}^{-1} = \frac{a\beta + e}{e}\underline{I} - \frac{a\alpha\beta}{e(b\alpha + e)}\underline{J} \tag{42a}$$

which leads to

$$\underline{Q}^{-1} = \frac{1}{a\beta + e}\left(e\underline{I} + \frac{a\alpha\beta}{a\beta + b\alpha + e}\underline{J}\right). \tag{42b}$$

(All matrices are of order b, and $\underline{J}^2 = b\underline{J}$.) Hence, from (34)

$$\begin{aligned}
(1/\beta)\underline{E}_{ii'} &= -(1/\beta)\underline{K}^{-1}\underline{Q}^{-1}\underline{K}^{-1} \\
&= \frac{-\beta}{e^2(a\beta+e)}\left(\underline{I} - \frac{\alpha}{b\alpha+e}\underline{J}\right)\left(e\underline{I} + \frac{a\alpha\beta}{a\beta+b\alpha+e}\underline{J}\right)\left(\underline{I} - \frac{\alpha}{b\alpha+e}\underline{J}\right) \\
&= \frac{-\beta}{e^2(a\beta+e)}\left[e\underline{I} + \frac{\alpha}{(b\alpha+e)(a\beta+b\alpha+e)}\underline{J}\left\{-e(a\beta+b\alpha+e) + a\beta(b\alpha+e) - ab\alpha\beta\right\}\right] \\
&\qquad \times \left(\underline{I} - \frac{\alpha}{b\alpha+e}\underline{J}\right)
\end{aligned}$$

$$= \frac{-\beta}{e(a\beta+e)} \left(\underline{I} - \frac{\alpha}{a\beta+b\alpha+e} \underline{J}\right)\left(\underline{I} - \frac{\alpha}{b\alpha+e} \underline{J}\right)$$

$$= \frac{-\beta}{e(a\beta+e)} \left[\underline{I} + \frac{\alpha}{(b\alpha+e)(a\beta+b\alpha+e)} \underline{J} \; (- a\beta - b\alpha - e - b\alpha - e + b\alpha)\right]$$

$$= \frac{-\beta}{e(a\beta+e)} \left[\underline{I} - \frac{\alpha(a\beta + b\alpha + 2e)}{(b\alpha+e)(a\beta+b\alpha+e)} \underline{J}\right], \quad \text{for } i \neq i' .$$

And from (33)

$$(1/\beta)\underline{E}_{ii} = (1/\beta)\underline{K}_i^{-1} + (1/\beta)\underline{E}_{ii'}$$

$$= \frac{1}{e}\left(\underline{I} - \frac{\alpha}{b\alpha + e} \underline{J}\right) + (1/\beta)\underline{E}_{ii'} .$$

Thus on writing

$$(1/\beta)\underline{E}_{ii'} = r\underline{I} + s\underline{J} ,$$

we have

$$(1/\beta)\underline{E}_{ii} = (r + \tfrac{1}{e})\underline{I} + \left[s - \frac{\alpha}{e(b\alpha + e)}\right] \underline{J} = p\underline{I} + q\underline{J}$$

where

$$r = \frac{-\beta}{e(a\beta + e)} \tag{43}$$

$$s = \frac{\alpha\beta(a\beta + b\alpha + 2e)}{e(a\beta + e)(b\alpha + e)(a\beta + b\alpha + e)} \tag{44}$$

$$p = r + \frac{1}{e} \tag{45}$$

$$q = s - \frac{\alpha}{e(b\alpha + e)} . \tag{46}$$

Then, from (32) and (42)

$$\underset{\sim}{V}^{-1} = \underset{\sim}{F}^{-1} = (1/\beta)\{\underset{\sim}{E}_{ii'}\} \quad \text{for } i,i' = 1, 2, \cdots, a$$

$$= \begin{bmatrix} p\underset{\sim}{I} + q\underset{\sim}{J} & r\underset{\sim}{I} + s\underset{\sim}{J} & r\underset{\sim}{I} + s\underset{\sim}{J} & \cdots \\ r\underset{\sim}{I} + s\underset{\sim}{J} & p\underset{\sim}{I} + s\underset{\sim}{J} & r\underset{\sim}{I} + s\underset{\sim}{J} & \cdots \\ \vdots & \vdots & p\underset{\sim}{I} + s\underset{\sim}{J} & \\ & & & \ddots \end{bmatrix}_{a \times a}$$

$$= (p - r)\underset{\sim}{I}_{ab} + (q - s)\begin{bmatrix} \underset{\sim}{J}_b & & \underset{\sim}{0} \\ & \ddots & \\ \underset{\sim}{0} & & \underset{\sim}{J}_b \end{bmatrix} + r\begin{bmatrix} \underset{\sim}{I}_b & \cdots & \underset{\sim}{I}_b \\ \vdots & & \vdots \\ \underset{\sim}{I}_b & \cdots & \underset{\sim}{I}_b \end{bmatrix} + s\underset{\sim}{J}_{ab} \qquad (47)$$

where from (43) - (46),

$$p - r = \frac{1}{e}$$

$$q - s = \frac{-\alpha}{e(b\alpha + e)} \qquad (48)$$

$$r \quad = \text{as in } (43)$$

and

$$s \quad = \text{as in } (44)\ .$$

These are exactly the results given by Wallace and Hussain (1969), in their equations (10) - (13), where they have written $\underset{\sim}{V}$ (for $n_{ij} = 1$ and $\gamma = 0$) as

$$\underset{\sim}{V} = e\underset{\sim}{I} + \alpha\begin{bmatrix} \underset{\sim}{J}_b & & \underset{\sim}{0} \\ & \ddots & \\ \underset{\sim}{0} & & \underset{\sim}{J}_b \end{bmatrix} + \beta\begin{bmatrix} \underset{\sim}{I}_b & \cdots & \underset{\sim}{I}_b \\ \vdots & & \vdots \\ \underset{\sim}{I}_b & \cdots & \underset{\sim}{I}_b \end{bmatrix}\ .$$

5. Special Cases

Inversion of (41), namely of

$$\underset{\sim}{Q} = \text{diag}\left\{1 + \sum_{i=1}^{a} 1/d_{i1} \cdots 1 + \sum_{i=1}^{a} 1/d_{ib}\right\} - \lambda \sum_{i=1}^{a} \frac{1}{1 + \lambda \sum_{j=1}^{b} 1/d_{ij}} \underset{\sim}{r}_i \underset{\sim}{r}_i' \qquad (41)$$

where $\lambda = \alpha/\beta$ and

$$\underset{\sim}{r}_i' = [1/d_{i1} \cdots 1/d_{ib}] \quad \text{and} \quad 1/d_{ij} = \frac{n_{ij}\beta}{n_{ij}\gamma + e} \qquad (49)$$

is, "obviously", not going to be easy---not even for special cases; e.g., not even for proportional subclasses will it be easy, because of the manner of occurrence of the n_{ij}'s in $1/d_{ij}$. One case is, however, somewhat amenable, namely the no-interaction case, for which $\gamma = 0$.

a. No interaction

With $\gamma = 0$ we have $1/d_{ij} = n_{ij}\beta/e$ and so

$$\sum_{i=1}^{a} 1/d_{ij} = n_{.j}\beta/e \quad \text{and} \quad \sum_{j=1}^{b} 1/d_{ij} = n_{i.}\beta/e,$$

and

$$\underset{\sim}{r}_i' = \beta/e(n_{i1}\ n_{i2} \cdots n_{ib}) \ . \qquad (50)$$

Then (41) becomes

$$\underset{\sim}{Q} = \text{diag}\left\{1 + n_{.1}\beta/e \cdots 1 + n_{.b}\beta/e\right\} - \lambda e \sum_{i=1} \frac{1}{e + n_{i.}\lambda\beta} \underset{\sim}{r}_i \underset{\sim}{r}_i' \ . \qquad (51)$$

The $\underset{\sim}{r}_i \underset{\sim}{r}_i'$ under the summation in (51) does, for $\underset{\sim}{r}_i$ of (50), appear to be intractable.

b. Proportional subclasses, no interaction

Suppose $n_{ij} = k_i p_j$ for all i and j. Then with

$$\underline{p}' = (p_1\ p_2\ \cdots\ p_b) \tag{52}$$

we have from (50)

$$\underline{r}_i' = (k_i\beta/e)\underline{p}\ . \tag{53}$$

Hence (51) becomes

$$\underline{Q} = \text{diag}\left\{1 + n_{.j}\beta/e\right\} - \left(\alpha\beta/e \sum_{i=1}^{a} \frac{k_i^2}{e + n_{i.}\alpha}\right)\underline{p}\underline{p}' \quad \text{for } j = 1 \cdots b$$

$$= \underline{\Delta} - \tau\underline{p}\underline{p}' \tag{54}$$

where $\underline{\Delta}$ is the diagonal matrix

$$\underline{\Delta} = \text{diag}\left\{1 + n_{.j}\beta/e\right\} \quad \text{for } j = 1, 2, \cdots, b \tag{55}$$

and τ is the scalar

$$\tau = \sum_{i=1}^{a} \frac{k_i^2\alpha\beta/e}{e + n_{i.}\alpha}\ . \tag{56}$$

Then from (37)

$$\underline{Q}^{-1} = \underline{\Delta}^{-1} + \frac{\tau}{1 - \tau\underline{p}'\underline{\Delta}^{-1}\underline{p}}\,\underline{\Delta}^{-1}\underline{p}\underline{p}'\underline{\Delta}^{-1}\ , \tag{57}$$

and from (52), (55), and (56) this is

$$\underline{Q}^{-1} = \text{diag}\left\{\frac{e}{e + n_{.j}\beta}\right\} + \frac{\tau}{1 - \tau \sum_{j=1}^{b} \frac{p_j^2 e}{e + n_{.j}\beta}} \left\{\frac{p_j p_{j'} e^2}{(e + n_{.j}\beta)(e + n_{.j'}\beta)}\right\} \tag{58}$$

for $j, j' = 1, 2, \cdots, b$.

Also, from (40)

$$\underline{K}_i^{-1} = \text{diag}\left\{1/d_{i1} \cdots 1/d_{ib}\right\} - \frac{\lambda(\beta k_i/e)^2}{1 + \lambda(\beta k_i/e)\underline{p}'\underline{1}} \underline{p}\underline{p}'$$

$$= (\beta k_i/e)\underline{D}_p - \frac{(\alpha\beta/e)k_i^2}{e + \alpha k_i p_.} \underline{p}\underline{p}' \qquad (59)$$

where $\underline{D}_p$ is the diagonal matrix of elements in $\underline{p}$ of (52) and $p_. = \underline{p}'\underline{1} = \sum_{j=1}^{b} p_j$.

From (58) and (59) $\underline{E}_{ii}$ and $\underline{E}_{ii'}$ of (33) and (34) can be obtained and thence $\underline{F}^{-1}$ of (32), $\underline{H}$ of (31a), and $\underline{V}^{-1}$ of (18a) and (18b). By this means, using $\underline{V}^{-1}$ in (6) along with the derivatives in (7) ---ignoring that for γ---the elements of the information matrix of the variance components in the proportional subclass case can be obtained.

Incidentally, a check on $\underline{Q}$ and $\underline{Q}^{-1}$ of (54) and (58) is that for all $n_{ij} = 1$ (balanced data) they reduce to the results of the previous section. For, with $n_{ij} = 1$, we have from (52), (55), and (56)

$$k_i = 1, \quad p_i = 1, \quad \text{and} \quad \underline{p} = \underline{1} \; ;$$

$$\underline{\Delta} = (1 + a\beta/e)\underline{I} \; ,$$

and

$$\tau = a\alpha\beta/e(e + b\alpha) \; .$$

Hence, $\underline{Q}$ of (54) is

$$\underline{Q} = \frac{e + a\beta}{e}\underline{I} - \frac{a\alpha\beta}{e(e + b\alpha)}\underline{J} \quad \text{of (42a)}$$

and $\underline{Q}^{-1}$ of (58) is

$$\underset{\sim}{Q}^{-1} = \frac{e}{e + a\beta}\underset{\sim}{I} + \frac{a\alpha\beta}{[e(e + b\alpha) - a\alpha\beta be/(e + a\beta)]}\,\frac{e^2}{(e + a\beta)^2}\underset{\sim}{J}$$

$$= \frac{e}{e + a\beta}\underset{\sim}{I} + \frac{a\alpha\beta e^2}{e[(e + b\alpha)(e + a\beta) - a\alpha\beta b](e + a\beta)}\underset{\sim}{J}$$

$$= \frac{e}{e + a\beta}\underset{\sim}{I} + \frac{e}{e + a\beta}\,\frac{a\alpha\beta}{a\beta + b\alpha + e}\underset{\sim}{J} \quad \text{of (42b).}$$

6. Other Ideas

a. Another lexicon order

Throughout all of sections 1, 2, and 3 we have considered $\underset{\sim}{y}$ as the vector of observations ordered by k within j within i. The whole problem could also be treated by considering $\underset{\sim}{z}$ as the vector of observations y_{ijk} ordered by k within i within j. Clearly $\underset{\sim}{z} = \underset{\sim}{P}\underset{\sim}{y}$ where $\underset{\sim}{P}$ is a permutation matrix and $\text{var}(\underset{\sim}{z}) = \underset{\sim}{W}$, say, with $\underset{\sim}{W} = \underset{\sim}{P}\underset{\sim}{V}\underset{\sim}{P}'$. In this case the matrix corresponding to (41) will be

$$\underset{\sim}{Q}^* = \text{diag}\left\{1 + \sum_{j=1}^{b} 1/f_{1j} \cdots 1 + \sum_{j=1}^{b} 1/f_{aj}\right\} - \theta \sum_{j=1}^{b} \frac{1}{1 + \theta \sum_{i=1}^{a} 1/f_{ij}}\,\underset{\sim}{s}_j\underset{\sim}{s}_j'$$

where

$$f_{ij} = (n_{ij}\gamma + e)/n_{ij}\alpha$$

$$\theta = 1/\lambda = \beta/\alpha$$

and

$$\underset{\sim}{s}_j' = (1/f_{1j} \;\; 1/f_{2j} \cdots 1/f_{aj}).$$

Both approaches, using $\underset{\sim}{y}$ and $\underset{\sim}{V}$ or $\underset{\sim}{z}$ and $\underset{\sim}{W}$, must yield the same values for the elements of the information matrix. The hope here is that the use of $\underset{\sim}{z}$ and $\underset{\sim}{W}$ might, in conjunction with $\underset{\sim}{y}$ and $\underset{\sim}{V}$, lead to deriving $\underset{\sim}{V}^{-1}$. (We believe it is a feint hope!)

b. A recurrence procedure

In (41) define

$$\varphi_i^2 \equiv \frac{\lambda}{1 + \lambda \sum_{j=1}^{b} 1/d_{ij}} ,$$

$$\underline{t}_i \equiv \varphi_i \underline{r}_i ,$$

and

$$\underline{\Delta} = \text{diag}\left\{1 + \sum_{i=1}^{a} 1/d_{ij}\right\} \quad \text{for } j = 1, 2, \cdots, b.$$

Then

$$\underline{Q} = \underline{\Delta} - \sum_{i=1}^{a} \underline{t}_i \underline{t}_i' .$$

Now define

$$\underline{C}_1 = \underline{\Delta} - \underline{t}_1 \underline{t}_1' ,$$

$$\underline{C}_2 = \underline{C}_1 - \underline{t}_2 \underline{t}_2' ,$$

$$\vdots$$

and $\underline{C}_i = \underline{C}_{i-1} - \underline{t}_i \underline{t}_i' , \quad \text{for } i = 2 \cdots a.$

Then $\underline{Q} = \underline{C}_a$

and repetitive use of (37) yields the inverses of the $\underline{C}$'s. Thus

$$\underline{C}_1^{-1} = \underline{\Delta}^{-1} + \frac{1}{1 - \underline{t}_1' \underline{\Delta}^{-1} \underline{t}_1} (\underline{\Delta}^{-1} \underline{t}_1 \underline{t}_1' \underline{\Delta}^{-1})$$

$$\underline{C}_2^{-1} = \underline{C}_1^{-1} + \frac{1}{1 - \underline{t}_2' \underline{C}_1^{-1} \underline{t}_2} (\underline{C}_1^{-1} \underline{t}_2 \underline{t}_2' \underline{C}_1^{-1})$$

$$\vdots$$

and $\underline{C}_i^{-1} = \underline{C}_{i-1}^{-1} + \dfrac{1}{1 - \underline{t}_i' \underline{C}_{i-1}^{-1} \underline{t}_i} (\underline{C}_{i-1}^{-1} \underline{t}_i \underline{t}_i' \underline{C}_{i-1}^{-1}) \quad \text{for } i = 2, \cdots, a,$

with $\underline{Q}^{-1} = \underline{C}_a^{-1}$.

Unfortunately this process does not appear to yield analytically tractable results in the general case.

This recurrence procedure is given in Ralston (1965, pp. 462-463).

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