A SUPPLEMENT TO
THE RATIO OF AVERAGES AND THE AVERAGE OF RATIOS
AS 'BEST' ESTIMATES IN REGRESSION

by

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The paper entitled "The Ratio of Averages and the Average of Ratios as 'Best' Estimates in Linear Regression" considered the following problem:

Given \( n \) pairs of observations \((X_1,Y_1), (X_2,Y_2), \ldots, (X_n,Y_n)\)
where \( X_1, X_2, \ldots, X_n \) are known non-negative constants, \( Y_1, Y_2, \ldots, Y_n \) are independent chance variables with
\[
Y_i = \beta X_i + \epsilon_i \quad \text{for } i = 1, 2, \ldots, n;
\]
\( \beta \) and \( \mathbb{E}(\epsilon_i^2) = \sigma_i^2 \) are unknown, find the minimum variance linear unbiased estimator of \( \beta \) for the cases

I. \( \sigma_i^2 = kX_i \), where \( k \) is an unknown constant
II. \( \sigma_i^2 = cX_i^2 \), where \( c \) is an unknown constant.

We now consider a generalization of this problem by introducing a second parameter \( \alpha \), an "intercept parameter", into the linear model; i.e., we take as before \( n \) independent pairs of observations \((X_i,Y_i)\) where \( X_1, X_2, \ldots, X_n \) are known non-negative constants and \( Y_i = \alpha + \beta X_i + \epsilon_i \) for \( i = 1, 2, \ldots, n; \)
\( \alpha, \beta \) and \( \mathbb{E}(\epsilon_i^2) = \sigma_i^2 \) are unknown and find the minimum variance linear unbiased estimators of \( \alpha \) and \( \beta \) for the cases I and II.

The estimator \( b \) of \( \beta \) is to be linear in \( (Y_1, Y_2, \ldots, Y_n) \);
i.e., it is to be of the form \( b = \sum_{i=1}^{n} u_i Y_i \) where \( u_1, u_2, \ldots, u_n \) are constants not depending on \( Y_1, Y_2, \ldots, Y_n \). Further, we have stipulated that the estimator \( b \) must be unbiased, i.e.,
$E(b) = \beta$. This last condition then implies that we must choose the constants $u_1, u_2, \ldots, u_n$ so that $\sum_{i=1}^{n} u_i = 0$ and $\sum_{i=1}^{n} u_i X_i = 1$, since $E(b) = E(\sum_{i=1}^{n} u_i Y_i) = \sum_{i=1}^{n} E(u_i) = \sum_{i=1}^{n} a + \beta X_i = \alpha \sum_{i=1}^{n} u_i + \beta \sum_{i=1}^{n} u_i X_i = \beta$.

Finally, we imposed the restriction that from among all possible sets of constants $u_1, u_2, \ldots, u_n$ which satisfy the conditions $\sum_{i=1}^{n} u_i = 0$, $\sum_{i=1}^{n} u_i X_i = 1$ we must choose that set which gives us the estimator having the smallest variance, i.e., we choose $u_1, u_2, \ldots, u_n$ so that $\text{var}(b) = \text{var}(\sum_{i=1}^{n} u_i Y_i) = \sum_{i=1}^{n} \text{var}(u_i) = \sum_{i=1}^{n} \sigma_i^2$ is as small as possible. The minimization of $\text{var}(b)$ with respect to $u_1, u_2, \ldots, u_n$, subject to the restrictions $\sum_{i=1}^{n} u_i = 0$, $\sum_{i=1}^{n} u_i X_i = 1$ is performed as follows:

$$F(u_1, u_2, \ldots, u_n) = \sum_{i=1}^{n} \sigma_i^2 - 2 \mu (\sum_{i=1}^{n} u_i) - 2 \lambda (\sum_{i=1}^{n} u_i X_i - 1)$$

$$\frac{\delta F(u_1, u_2, \ldots, u_n)}{\delta u_i} = 2u_i \sigma_i^2 - 2 \mu - 2 \lambda X_i = 0$$

$$u_i = \frac{\mu + \lambda X_i}{\sigma_i^2}$$

or, writing $w_i = \frac{1}{\sigma_i^2}$,

(1) $$u_i = w_i (\mu + \lambda X_i)$$

multiplying both sides of (1) by $X_i$ we get

(2) $$u_i X_i = w_i X_i (\mu + \lambda X_i)$$

Summing (1) for $i = 1, 2, \ldots, n$ we get

(3) $$\sum_{i=1}^{n} u_i = u \sum_{i=1}^{n} w_i + \lambda \sum_{i=1}^{n} w_i X_i$$

and since $\sum_{i=1}^{n} u_i = 0$, we get

$$\mu = - \lambda \frac{\sum_{i=1}^{n} w_i X_i}{\sum_{i=1}^{n} w_i}$$

or, denoting the weighted mean $\frac{\sum_{i=1}^{n} w_i X_i}{\sum_{i=1}^{n} w_i} = \bar{x}_w$, we have

(4) $$\mu = - \lambda \bar{x}_w$$
Summing (2) for \( i = 1, 2, \ldots, n \) we get
\[
\sum u_i x_i = \mu \Sigma w_i x_i^2 + \lambda \Sigma w_i x_i^2
\]
and since \( \sum u_i x_i = 1 \), we get from (4) and (5)
\[
\lambda = \frac{1}{\Sigma w_i (x_i - \bar{x}_w)^2}
\]
or,
\[
(6) \quad \lambda = \frac{1}{\Sigma w_i (x_i - \bar{x}_w)^2}
\]
Upon substituting (4) and (6) into (1) we have
\[
u_i = \frac{w_i (x_i - \bar{x}_w)}{\Sigma w_i (x_i - \bar{x}_w)^2}
\]
Hence,
\[
b = \sum u_i y_i = \frac{\Sigma w_i (x_i - \bar{x}_w) y_i}{\Sigma w_i (x_i - \bar{x}_w)^2}
\]
which for case I \((w_i = \frac{1}{k X_i})\) gives
\[
b = \frac{X_i Y_i}{\Sigma \frac{1}{k X_i}} = \frac{\Sigma Y_i}{\Sigma \frac{1}{k X_i}} - \frac{n}{\Sigma \frac{1}{k X_i}}
\]
and
\[
b = \frac{X_i^2}{\Sigma \frac{1}{k X_i}} - \frac{\Sigma \frac{1}{k X_i}}{\Sigma \frac{1}{k X_i}} x_i = \frac{\Sigma X_i}{\Sigma \frac{1}{k X_i}} - \frac{n^2}{\Sigma \frac{1}{k X_i}}
\]
\[
= \bar{y} - \bar{y}_w = \frac{\bar{y} - \bar{y}_w}{\bar{x} - \bar{x}_w}
\]
where

\[
\tilde{\bar{y}}_w = \frac{\Sigma w_i y_i}{\Sigma w_i} = \frac{\frac{y_i}{kX_i}}{\frac{1}{kX_i}} = \frac{\frac{y_i}{X_i}}{\Sigma X_i}
\]

\[
\tilde{\bar{x}}_w = \frac{\Sigma w_i x_i}{\Sigma w_i} = \frac{\frac{x_i}{kX_i}}{\frac{1}{kX_i}} = \frac{n}{\Sigma X_i}
\]

Similarly, it may be shown that the minimum variance linear unbiased estimator \( a \) of \( \alpha \) is for this case

\[
a_I = \tilde{\bar{y}}_w - b_I \tilde{\bar{x}}_w.
\]

The estimator \( b_{II} \) for the second case, where \( w_i = \frac{1}{cX_i^2} \) is

\[
b_{II} = \frac{\frac{1}{X_i} \frac{\Sigma X_i^2}{X_i}}{\frac{1}{X_i} \frac{\Sigma X_i^2}{X_i}} \frac{\Sigma X_i}{n - \frac{\Sigma X_i^2}{X_i}}
\]

and

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Next, we observe that the variances of our estimates are, in general,
\[ \text{var}(b) = \sum u_i^2 \sigma_i^2 = \frac{1}{\sum w_i (X_i - \bar{X}_w)^2} \]
\[ \text{var}(a) = \frac{1}{\sum w_i} + \frac{\bar{X}_w^2}{\sum w(X_i - \bar{X}_w)^2} \]
which for our two cases reduce to
\[ \text{var}(b_I) = \frac{k}{n(\bar{x} - \bar{x}_w)} \]
\[ \text{var}(a_I) = \frac{k}{n} \left( \frac{\bar{X}_w}{\bar{x}_w} + \frac{\bar{x}_w}{\bar{x} - \bar{x}_w} \right) \]
\[ \text{var}(b_{II}) = \frac{c}{(\sum \frac{1}{X_i})^2} \]
\[ \frac{n - \frac{1}{\Sigma X_i^2}}{\Sigma X_i^2} \]
\[ \text{var}(a_{II}) = \frac{c}{\Sigma \frac{1}{X_i^2} - \frac{1}{n}(\Sigma \frac{1}{X_i})^2} \]
Unbiased estimates of the above variances can be calculated from the general formulas:

\[ \hat{\sigma}_{b}^2 = \frac{1}{n-2} \frac{\sum w_1 (Y_1 - \bar{y}_w)^2}{\sum w_1 (X_1 - \bar{x}_w)^2} - b^2 \]

\[ \hat{\sigma}_{d}^2 = \frac{\sum w_1 x_1^2}{\sum w_1} \hat{\sigma}_{b}^2 \]

since

\[ \mathbb{E}(\hat{\sigma}_{b}^2) = \frac{\mathbb{E} \sum w_1 (a + \beta X_1 + \varepsilon_1 - a - \beta \bar{x}_w - \frac{\sum w_1 \varepsilon_1}{\sum w_1})^2 - \mathbb{E} \left[ \sum w_1 (a + \beta X_1 + \varepsilon_1) \right]^2}{(n-2) \sum w_1 (X_1 - \bar{x}_w)^2} - \frac{\mathbb{E} \left[ \sum w_1 (a + \beta X_1 + \varepsilon_1) \right]^2}{(n-2)} \]

\[ = \frac{\beta^2 + \frac{\sum w_1 \sigma_1^2}{\sum w_1} - \frac{\sum w_1^2 \sigma_1^2}{\sum w_1}}{n-2} + \frac{\beta^2}{(n-2) \sum w_1 (X_1 - \bar{x}_w)^2} - \frac{\sum w_1^2 \sigma_1^2}{n-2} - \frac{\sum w_1^2 \sigma_1^2}{n-2} \]

\[ = \frac{n-1}{(n-2) \sum w_1 (X_1 - \bar{x}_w)^2} - \frac{1}{(n-2) \sum w_1 (X_1 - \bar{x}_w)^2} \]

\[ = \frac{1}{\sum w_1 (X_1 - \bar{x}_w)^2} \]

For cases I and II these estimates of variance reduce to

\[ \hat{\sigma}_{b}^2 = \left[ \frac{\sum w_1^2}{n \bar{x} - \bar{x}_w} \right] - b^2 \]
\[
\sigma_{a_{II}}^2 = \frac{\sum X_i}{\sum X_i^2} \sigma_{b_{II}}^2 \\
\sigma_{b_{II}}^2 = \left[ \frac{\frac{1}{\sum X_i^2} - \frac{1}{\sum X_i} \frac{(\sum Y_i)^2}{\sum X_i^2}}{n - 1} \right] \frac{1}{n-2} \\
\sigma_{a_{II}}^2 = \frac{n}{\sum \frac{1}{X_i^2}} \sigma_{b_{II}}^2.
\]