Experimental Designs and Combinatorial Systems Associated with Latin Squares and Sets of Mutually Orthogonal Latin Squares[1]

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Abstract

In this expository paper we have demonstrated the importance of the theory of Latin squares and mutually orthogonal Latin squares in the field of design of experiments and combinatorial analysis. It is shown that many well-known and important designs and/or combinatorial systems are either equivalent or can be derived from Latin squares or a set of mutually orthogonal Latin squares.

I. Introduction. A Latin square of order n on a set Σ containing n distinct elements is an $n \times n$ matrix each of whose rows and columns is a permutation of the set Σ. Two Latin squares of order n are said to be orthogonal if, when they are superposed, each symbol of the first square occurs just once with each symbol of the second square. A set of t mutually orthogonal Latin squares of order n is a set of t Latin squares of order n any two of which are orthogonal. Any such set will be denoted by $O(n,t)$.

Let $N(n)$ denote the maximum number of mutually orthogonal Latin squares of order n. Then it is known that (i) $N(n) \le n-1$ for all n. (ii) $N(n) \ge 2$ for all n except n=2,6. (iii) $N(n) \ge 3$ for $n > 51$, $N(n) \ge 5$ for $n > 62$, $N(n) \ge 6$ for $n > 90$ and $N(n) \ge 29$ for $n > 34{,}115{,}553$. (iv) $N(n) = n-1$ if n is

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a prime power. (v) $N(mn) \geq \min\{N(m),N(n)\}$. (vi) Let $d = n-1-r$ and if $n > \frac{1}{2}(d^4-2d^3+2d^2+d-2)$, then the existence of an $O(n,r)$ implies that $N(n) = n-1$, for example the existence of an $O(n,n-3)$, $n > 4$ implies the existence of an $O(n,n-1)$. (vii) If $n \equiv 1$ or $2 \pmod 4$, and $n = x^2+y^2$ has no solution in rational integers, then $N(n) < n-3$, for example if $n \equiv 6 \pmod 8$ then $N(n) < n-3$. (viii) $N(n) \to \infty$ as $n \to \infty$. Indeed it is shown that $N(n) \geq n^{1/17}-2$.

The theory of Latin squares and mutually orthogonal Latin squares owes its importance to the fact that many well-known designs and/or combinatorial systems are either equivalent or can be derived from Latin squares or sets of mutually orthogonal Latin squares; viz., balanced incomplete block designs, partially balanced incomplete block designs, group devisible designs, F-square designs, lattice designs, balanced weighing designs, orthogonal arrays, Hadamard matrices, affine planes, projective planes, partial geometries, nets, error correcting codes, an arrangement of non-attacking rooks on hyperdimensional chess board, and strongly regular graphs. These designs and/or combinatorial systems are unquestionably potent and important in all branches of design of experiments and combinatorial mathematics. The purpose of this paper is to indicate the relation of Latin squares and sets of mutually orthogonal Latin squares with the above designs and combinatorial systems.

II. Designs and Combinatorial Systems Associated with Latin Squares and Sets of Mutually Orthogonal Latin Squares.- In the following, for the sake of completeness, we shall define all the designs and combinatorial systems under our consideration.

Definition 2.1. A balanced incomplete block design with parameters v, b, r, k and λ is an arrangement of v symbols (treatments) in b subsets (blocks) of k symbols ($k \leq v$) satisfying the condition that any two distinct treatments occur together in λ blocks. Then any treatment occurs in r blocks and

$$vr = bk \ , \quad \lambda(v-1) = r(k-1) \ , \quad b \geq v \ .$$

We denote this design by BIB(v,b,r,k,λ). A BIB design is called *resolvable* if the blocks can be devided into r sets of v each such that in each set every symbol occurs exactly once. A BIB design is said to be *symmetrical* if $v = b$ and thus $r = k$ and hence will be denoted by BIB(v,k,λ). Some authors call a BIB design a 2-design or a BIB(v,k,λ) a (v,k,λ) configuration.

From any BIB(v,b,r,k,λ) one can form another design, called the complementary of the original, by putting in the i^{th} block of the complementary, those treatments which do not occur in the i^{th} block of the original. It is easy to see that the complementary design is a BIB$(v,b,b-r,v-k,b-2r+\lambda)$. From any BIB$(v,k,\lambda)$ we can obtain two other designs the derived and the residual designs. The derived design is obtained by omitting a block and retaining only the treatments of the omitted block in the remaining blocks. The derived design is a BIB$(k,v-1,k-1,\lambda,\lambda-1)$. In the residual design we omit a block and in the remaining blocks retain only those treatments which do not occur in the omitted block. The residual design is a BIB$(v-k,v-1,k,k-\lambda,\lambda)$.

Theorem 2.1. *(i) A Latin square of order 6 implies a* BIB$(36,15,6)$. *(ii) Existence of an* $O(2t,t-2)$ *implies the existence of a* BIB$(4t^2-1,2t^2-1,t^2-1)$ *and a* BIB$(4t^2,2t^2-t,t^2-t)$. *(iii) Existence of an* $O(2t-1,t-2)$ *implies the existence of a* BIB$\left((2t-1)^2,2(2t-1)^2,4t(t-1),2t(t-1),2t^2-t-1\right)$. *(iv) Existence of an* $O(n,n-1)$ *is equivalent to the existence of a* BIB$(n^2+n+1,n+1,1)$ *and a resolvable* BIB$(n^2,n^2+n,n+1,n,1)$.

Definition 2.2. A partially balanced incomplete block design (with two associate classes) with parameters v, b, r, k and λ_i, n_i, i=1,2 is an arrangement of v distinct objects (treatments) into b subsets (blocks) such that each contains exactly k distinct objects having the following properties: each object occurs in exactly r subsets, with respect to any specified object the remaining v-1 objects can be classified into 2 disjoint sets G_1 and G_2 of size n_1 and n_2 such that each object in G_i appears with the specified element in λ_i subsets and n_i being the same regardless of the element specified. If we call the elements that appear in a set λ_i times with a specified element θ, the i^{th} associate of θ, the number of elements common to the i^{th} associate θ and the j^{th} associate of ϕ, where θ and ϕ are k^{th} associates, is p_{ij}^k, this number being the same for any pair of k^{th} associates. The fundamental relations for such a design are:

$$n_1 + n_2 = v \ , \quad \lambda_1 n_1 + \lambda_2 n_2 = r(k-1) \quad .$$

$$p_{ij}^k = p_{ji}^k \ , \quad n_k p_{ij}^k = n_i p_{jk}^i = n_j p_{ik}^j$$

It is known that it is unnecessary to assume the constancy of all the p_{jk}^i's. If we assume that n_1, n_2, p_{11}^1 and p_{11}^2 are constant then the constancy of the rest follows. We shall denote this design by PBIB(v,b,r,k; λ_1,λ_2).

Theorem 2.2. (i) A Latin square of order n is equivalent to a PBPB($3n,n^2,n,3;0,1$). (ii) Existence of an O(n,t) is equivalent to the existence of a PBIB$\left(n(t+2),n^2,n,t+2;0,1\right)$. (iii) Existence of an O(n,t-2) is equivalent to the existence of a PBIB($n^2,tn,t,n;1,0$).

Definition 2.3. A group devisible design with parameters $v = mn, b, r, k, \lambda_1$ and λ_2 is a PBIB$(v, b, r, k; \lambda_1, \lambda_2)$ in which the v treatments partitioned into m disjoint sets of n each; such that any two treatments from the same set are 1-associates whereas any two treatments from different sets are 2-associates. This design is denoted by GD$(mn, b, r, k; \lambda_1, \lambda_2)$. A group divisible design with parameters mn, b, r, k, λ_1 and λ_2 is called a semi-regular group devisible design if $rk - \lambda_2 mn = 0$.

Theorem 2.3. (i) A Latin square of order n is equivalent to a semi-regular GD$(3n, n^2, n, 3; 0, 1)$. (ii) Existence of an $O(n,t)$ is equivalent to the existence of a semi-regular GD$\left((t+2)n, n^2, n, t+2; 0, 1\right)$.

Definition 2.4. Let $A = [a_{ij}]$ be an $n \times n$ matrix and let $\Sigma = \{c_1, c_2, \cdots, c_m\}$ be the ordered set of distinct elements of A. In addition, suppose that for each $k=1,2,\cdots,m$, c_k appears precisely λ_k times $(\lambda_k \geq 1)$ in eahc row and in each column of A. Then, A will be called a frequency square or, more concisely, an F-square design on Σ of order n and frequency vector $(\lambda_1, \lambda_2, \cdots, \lambda_m)$. We denote such a square by $F(n; \lambda_1, \lambda_2, \cdots, \lambda_m)$. Given an $F_1(n; \lambda_1, \cdots, \lambda_k)$ on a k-set $\Sigma = \{a_1, a_2, \cdots, a_k\}$ and an F-square $F_2(n; u_1, u_2, \cdots, u_t)$ on a t-set $\Omega = \{b_1, b_2, \cdots, b_t\}$. Then we say F_2 is an orthogonal mate for F_1, or F_1 and F_2 are orthogonal, if upon superposition of F_2 on F_1, a_i with frequency λ_i in F_1 appears $\lambda_i u_j$ times with b_j with frequency u_j in F_2. Let S_i be an n_i-set, $i=1,2,\cdots,t$. Let F_i be an F-square of order n and frequency vector $\bar{\lambda}_i = (\lambda_{i1}, \lambda_{i2}, \cdots, \lambda_{i\ell_i})$ on the set S_i. Then, we say $\{F_1, F_2, \cdots, F_t\}$ is a set of t mutually orthogonal F-squares if F_i is orthogonal to F_j, $i \neq j$, $i,j=1,2,\cdots,t$.

<u>Theorem 2.4</u>. <u>(i) A Latin square of order</u> n <u>implies an</u> $F(n;\bar{\lambda})$ <u>where</u> $\bar{\lambda}$ <u>is any vector of decomposition of</u> n. <u>(ii) Existence of an</u> $O(n,t)$ <u>implies the existence of a set of</u> t <u>mutually orthogonal F-squares</u> $F_i(n;\bar{\lambda}_i)$, $i=1,2,\cdots,t$ <u>where</u> $\bar{\lambda}_i$ <u>is any vector of decomposition of</u> n.

<u>Definition 2.5</u>. A one restrictional, two dimensional lattice design with parameters v and r is a collection of r classes of incomplete blocks each of size $\sqrt{v}$ filled out with v treatments such that (i) each class contains $\sqrt{v}$ blocks; (ii) if two blocks B and B' belong to two different classes, then they have exactly one treatment in common. This design is denoted by ORTD(v,r). A balanced one restrictional, two dimensional lattice design with parameters v, r and λ, denoted by BORTD(v,r,λ), is an ORTD(v,r) design with an additional property that every treatment appears with every other treatment, in the whole design, λ times.

<u>Theorem 2.5</u>. <u>(i) A Latin square of order</u> n <u>is equivalent to an</u> ORTD(n,3). <u>(ii) Existence of an</u> $O(n,t)$ <u>is equivalent to the existence of an</u> ORTD(n,t+2). <u>(iii) Existence of an</u> $O(n,n-1)$ <u>is equivalent to the existence of a</u> BORTD(n,n+1,1).

<u>Definition 2.6</u>. A balanced weighing design with parameters v, b, r, p, λ_1, and λ_2 (also called tournament design) is an arrangement of v objects in b blocks of size 2p, each block consisting of two half-blocks of size p. Two objects appear in the same half-block λ_1 times, and in the complementary half-blocks of the same block λ_2 times. r is the number of blocks in which each object appears. It is not difficult to see that $\lambda_1(v-1) = r(p-1)$, $\lambda_2(v-1) = rp$. Thus $r = (\lambda_2-\lambda_1)(v-1)$ and $b = (\lambda_2-\lambda_1)v(v-1)/2p$. We denote this design by BWD$(v,b,r,p,\lambda_1,\lambda_2)$.

Theorem 2.6. (i) Existence of an $O(2t,t-2)$ implies the existence of a $BWD(4t^2,4t^2-1,4t^2-1,t^2,t^2-1,t^2)$. (iii) Existence of an $O(n,n-1)$ implies the existence of a $BWD\left(n^2,n(n^2-1)/2,n^2-1,n,n-1,n\right)$.

Definition 2.7. A $k \times N$ matrix A, with entries from a set Σ of $s \geq 2$ elements, is called an orthogonal array of strength t, size N, depth (constraints) k, and s levels (orders), if each $t \times N$ submatrix of A contains all possible $t \times 1$ column vectors with the same frequency λ. It is clear that $N = \lambda s^2$. We shall denote such an array by $OA(N,k,s,t,\lambda)$.

Theorem 2.7. (i) A Latin square of order n is equivalent to an $OA(n^2,3,n,2,1)$. (ii) A Latin square of order 6 implies an $OA(36,35,2,2,9)$. (iii) Existence of an $O(n,t)$ is equivalent to the existence of an $OA(n^2,t+2,n,2,1)$. (iv) Existence of an $O(2r,r-2)$ implies the existence of an $OA(4r^2,4r^2-1,2,2,r^2)$.

Definition 2.8. A Hadamard matrix of order m, denoted by H(m), is an $m \times m$ matrix of +1,s and -1,s such that $H(m)H^t(m) = mI$, where $H^t(m)$ is the transpose of H(m) and I is the identity matrix of order m. This is equivalent to the assertion that any two rows of H(m) are orthogonal.

Theorem 2.8. (i) A Latin square of order 6 implies a H(36). (ii) Existence of an $O(2r,r-2)$ implies the existence of a $H(4r^2)$.

Definition 2.9. Consider a system containing a set of distinct elements called "points" and certain subsets of them called "lines" together with an incidence relation (a point incident with a line or a line incident with a point). The system of points and lines is said to form an affine plane (or Euclidean plane) if the following axioms are satisfied:

(1) Two distinct lines have at most one point in common.

(2) Two distinct points are incident with one and only one line.

(3) Given a line and a point not on it, there is a unique line parallel to the given line going through the given point.

(4) There are at least three noncollinear points in the plane.

The plane is said to be of order n ($n > 1$ by asiom 4) if some line in the plane has exactly n points. Then it is not difficult to show that all the lines contain exactly n points and that the plane contains exactly n^2 distinct points and n^2+n distinct lines. We shall denote any such plane by AF(2,n).

<u>Theorem 2.9.</u> <u>Existence of an</u> O(n,n-1) <u>is equivalent to the existence of an</u> AF(2,n).

<u>Definition 2.10.</u> A system of points and lines is said to be a projective plane if the following axioms are satisfied:

(1) Two distinct lines have exactly one point in common.

(2) Two distinct points are incident with one and only one line.

(3) There are at least four points in the system, no three of which are collinear.

The plane is said to be of order n if some lines in the plane have exactly n points. It is not difficult to show that a finite projective plane of order n has exactly n^2+n+1 points and n^2+n+1 lines and that every line contains exactly n+1 points. Such a plane is denoted by $\Pi(2,n)$.

<u>Theorem 2.10.</u> <u>Existence of an</u> O(n,n-1) <u>is equivalent to the existence of a</u> $\Pi(2,n)$.

Definition 2.11. A partial geometry with parameters r, k and t, denoted by PG(r,k,t), is a system of points and lines, and a relation of incidence which satisfies the following axioms:

(1) Any two distinct points are incident with not more than one line.

(2) Each point is incident with r lines.

(3) Each line is incident with k points.

(4) If the point p is not incident with the line ℓ, then there are exactly t lines ($t \geq 1$) which are incident with p and also incident with some points incident with ℓ.

Clearly, we have $1 \leq t \leq k$, $1 \leq t \leq r$, where r and k are ≤ 2. It is easily seen from an examination of the four axioms above that given a PG(r,k,t), we can obtain a dual PG(k,r,t) by changing points to lines and lines to points. The number of points v and the number of lines b in a PG(r,k,t) satisfy the relations

$$v = k[(r-1)(k-1)+t]/t$$

$$b = r[(r-1)(k-1)+t]/t \quad .$$

Theorem 2.11. (i) A Latin square of order n is equivalent to a PG(3,n,2). (ii) Existence of an O(n,t-2) is equivalent to the existence of a PG(t,n,t-1).

Definition 2.12. Let k and n be two positive integers, with $k \geq 3$. A finite net of degree k and order n, denoted by N(n,k), is a system of points and lines such that:

(1) N(n,k) contains k nonempty classes of parallel lines.

(2) Two lines belonging to distinct classes have a unique common point.

(3) Each point is on exactly one line of each class.

(4) Some lines of $N(n,k)$ have exactly n distinct points.

It can be shown that every line of $N(n,k)$ contains exactly n distinct points, every class of lines contains exactly n lines and finally such a system consists of n^2 distinct points, kn distinct lines and either $n = 1$ or $n \geq k-1$.

Theorem 2.12. (i) A Latin square of order n is equivalent to a $N(n,3)$. (ii) Existence of an $O(n,t)$ is equivalent to the existence of a $N(n,t+2)$.

Definition 2.13. Coding is the representation of information (signals, numbers, messages, etc.) by code symbols or sequences of code symbols (often called code words). The set of code words and their mapping, which determine the set, characterize a code. Information is said to be placed into code form by encoding and extracted from code form by decoding. Certain codes may have larger code length than others. Such codes are said to contain "redundancy" which can be used to advantage for error control. We are concerned here about certain codes called block codes. More formally, let Σ be a set of n distinct elements. Denote the set of all k tuples over Σ by Σ_n^k. Any subset of Σ_n^k may be termed a block code (the block length being k) the elements of the subset are termed code words. The Hamming distance between two code words is the number of components in which they disagree. With this definition of distance a block code satisfies the axioms for a metric space. A block code in which any pair of code words are at least a Hamming distance of r apart is called a distance r code. A distance r code is also called $(r-1)/2$-error correcting code, because if fewer than $(r-1)/2$ changes are made in the components of any code word it is still closer to its original form than to any of the other code words. Similarly a distance r code is termed $(r-1)$-error detecting. Thus a distance r code is capable of detecting up to (r-1) error and correcting up to (r-1)/2 errors. We denote an m-subset of distance r code over Σ_n^k by Code(m,k,r;n).

Theorem 2.13. (i) A Latin square of order n implies a Code(n^2,3,2;n). (ii) Existence of an O(n,t) is equivalent to the existence of a Code (n^2,t+2,t+1;n). (iii) Existence of an O(2t,t-2) implies the existence of a Code ($8t^2$,$4t^2$,$2t^2$;2).

Definition 2.14. Consider an nxn chessboard. An arrangement of two rooks on this board is said to be non-attacking rooks if they are not in the same row or column. An arrangement of r rooks on an nxn chessboard is said to be mutually non-attacking if any two rooks are non-attacking. A generalization of this concept is as follows. An arrangement of r rooks on a t-dimensional nxn chessboard is said to be mutually non-attacking rooks if they are mutually non-attacking with respect to each dimension. Such an arrangement is denoted by NAR(r,n^t).

Theorem 2.14. (i) A Latin square of order n is equivalent to a NAR(n^2,n^3). (ii) Existence of an O(n,r) is equivalent to the existence of a NAR(n^2,n^{r+2}).

Definition 2.15. A finite undirected graph of order n with no loops or multiple lines is called regular if each point is adjacent to n_1 other points and non-adjacent to n_2 other points, so $n = n_1 + n_2 + 1$. A regular graph with $n_1 \neq 0$ and $n_2 \neq 0$ (thereby excluding the totally disconnected and complete graphs) is called strongly regular provided any two adjacent points are both adjacent to exactly p_{11}^1 other points, and any two nonadjacent points are both adjacent to exactly p_{11}^2 other points. The numbers n, n_1, p_{11}^1 and p_{11}^2 are convenient parameters for a strongly regular graph. Thus we denote such a graph by SRG($n,n_1 n p_{11}^1,p_{11}^2$).

The concept of a strongly regular graph is isomorphic with an association scheme having two associate classes when treatments are identified with points, a pair of first associates are identified with a pair of adjacent points, and a pair of second associates are identified with a pair of nonadjacent points. It is also known that the existence of a BIB(v,b,r,k,1) implies the existence of a $SRG\left(b,k(r-1),(r-2)+(k-1)^2,k^2\right)$. These results together with pvevious ones lead to the following theorem

Theorem 2.15. (i) A Latin square of order n *implies a* $SRG(n^2,3n-3,n,6)$.
(ii) Existence of an $O(n,r-2)$ *implies the existence of a* $SRG(n^2,rn-r,n+r^2-3r,r^2-r)$.
(iii) Existence of an $O(n,n-1)$ *implies the existence of a* $SRG(n^2+n+1,n^2+n,n^2+n-1,n^2+2n+1)$.

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