An Algebraic Property of the Totally Symmetric Loops Associated With Kirkman-Steiner Triple Systems

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Abstract

The concept of an x-root of degree r in a loop of order n is introduced. It is shown that the totally symmetric loop of order n+1 derived from any Kirkman-Steiner triple system of order n admits a maximal identity-root. A statistical-combinatorial application of this algebraic property is then indicated. Finally, two open problems are also given.

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Introduction. A mathematical system consisting of an n-set Ω and a binary operation $*$ is said to form a loop of order n if the following axioms are satisfied:

(1) Ω contains an identity element e such that $x * e = e * x = x$ for every x in Ω.

(2) Any two of the elements in the equation $x * y = z$ uniquely determine the third.

Since the notation $x * z$ is too bulky we shall use, hereafter, the notation xy instead. A loop is said to be a totally symmetric loop if it also satisfies

(3) $xy = yx$ and $x(xy) = y$ for all x and y in Ω.

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In this paper, we shall introduce and study an algebraic property of totally symmetric loops of order $n \equiv 3 \pmod 6$. In the final part of this paper we shall indicate, briefly, a statistical-combinatorial application of this study. A few open questions are also stated.

We begin by introducing and reviewing certain concepts and results that will be relevant to our forthcoming results.

Definition 1. We say a loop $\mathcal{L}$ of order n accepts a $(k_1,k_2,\ldots,k_r)$ orthogonal partition if the n^2 cells in the Cayley table of $\mathcal{L}$ can be divided into r mutually disjoint exhaustive sets $S_1,S_2,\ldots,S_r$; in such a way that (1) S_i has k_i cells from each row and each column, (2) each element of $\mathcal{L}$ appears k_i times in the cells of S_i, (3) $k_1+k_2+\ldots+k_r=n$. In particular a set of S_t is called a transversal of $\mathcal{L}$ if $k_t=1$. If two transversals have no cells in common, they are said to be parallel; if they have exactly one cell in common, they are called orthogonal.

A set $\{t_1,t_2,\ldots,t_r\}$ mutually orthogonal transversals of $\mathcal{L}$ is said to be an x-root of degree r if these transversals are all sharing a unique cell containing the element x. Clearly any x-root of degree r occupies $r(n-1)+1$ cells of the Cayley table of a loop of order n. An x-root of degree r in the Cayley table of a loop of order n is said to be a maximal x-root if $r=n-2$. The following lemma justifies this terminology.

Lemma 1. *For any x-root of degree r in a loop of order n*, $r \leq n-2$.

Proof. Let the cell in the given x-root that contains the element x occur in row i and column j . Then the remaining 2n-2 cells of row i and column j, together with the n-1 other cells containing the element x, cannot be in the x-root. Thus there remains only $n^2 - 3n+3$ cells to accommodate the given x-root. However, as pointed out before, this x-root must occupy $r(n-1)+1$ cells. Hence $r \leq n-2$.

Definition 2. Let Σ be an n-set, $n \equiv 1,3 \pmod 6$. Then a Steiner triple system of order n on Σ is a collection of $n(n-1)/6$ unordered triples (x,y,z) with x,y,z in Σ, such that every pair of distinct elements of Σ belongs to exactly one triple. A triple system of order $n \equiv 3 \pmod 6$ is said to be a Kirkman-Steiner triple system of order n if it is a Steiner triple system with the following additional stipulation: the set of triples can be partitioned into $r = (n-1)/2$ disjoint classes such that the totality of elements in each class exhaust the set on which the system is defined.

While Reiss [9] has shown the sufficiency of $n \equiv 1,3 \pmod 6$ for the existence of a Steiner triple system of order n, Ray-Chaudhri and Wilson [8] have proved the sufficiency of $n \equiv 3 \pmod 6$ for the existence of a Kirkman-Steiner triple system of order n .

The coextensiveness of totally symmetric loops of order n+1 with Steiner triple systems of order n has been shown by Bruck [2] who proved the following theorem:

Theorem 1. *A totally symmetric loop of order n+1 exists if and only if there exists a Steiner triple system of order n .*

For the sake of clarity of later arguments, we shall sketch a proof of this theorem here.

Proof. Let A be a totally symmetric loop of order n+1 and let $H = A - \{e, \text{the identity element in } A\}$. Then the collection of all unordered triples (x,y,z) with x,y,z in H, such that $xy = z$, forms a Steiner triple system on H . Conversely, given a Steiner triple system of order n on an n-set W, we can then form a totally symmetric loop of order n+1 from these triples as follows: Define an operation o on the set $\mathcal{L}^* = W \cup \{e\}$ by: (1) $a \circ b = c$ if and only if (a,b,c) is in $\mathcal{L}^*$, (2) $e \circ a = a \circ e = a$, and (3) $a^2 = e^2 = e$ for all a in $\mathcal{L}^*$. Then $\mathcal{L}^*$ together with the binary operation o forms a totally symmetric loop of order n+1 .

Let Σ be an n-set, $n \equiv 3 \pmod 6$ and let $\mathcal{K}$ be a Kirkman-Steiner triple system on Σ . Let also $\mathcal{L}^*$ be the totally symmetric loop of order n+1 derived from $\mathcal{K}$. Denote the identity element in $\mathcal{L}^*$ by e . Partition $\mathcal{L}^*$ into $r = (n-1)/2$ disjoint classes C_i, $i = 1,2,\ldots,r$ as described in definition 2 . Then we have the following lemma.

<u>Lemma 2</u>. <u>C_i determines an e-root of degree 2 in the Cayley table of $\mathcal{L}^*$</u> .

<u>Proof</u>. Denote an arbitrary triple in C_i by (a_{ij}, b_{ij}, c_{ij}) , $j = 1,2,\ldots,n/3$. Identify three cells in the Cayley table of $\mathcal{L}^*$ by the 2-tuples (a_{ij}, b_{ij}) , (b_{ij}, c_{ij}) and (c_{ij}, a_{ij}) , the components of each 2-tuple being the row and column indicies respectively. Now let j run through all the n/3 triples in C_i . Then the corresponding $3 \times n/3 = n$ cells determined by the preceding rule, together with the cell corresponding to row and column indices (e,e), form a transversal for $\mathcal{L}^*$. Denote this transversal by t_{i1} . Another transveral t_{i2} is obtained by considering the cell (e,e) and the three cells in the Cayley table described by the 2-tuples (b_{ij}, a_{ij}), (c_{ij}, b_{ij}) and (a_{ij}, c_{ij}), where we let j run through the values $1,2,\ldots,n/3$. These exhibition rules clearly guarantee that t_{i1} is orthogonal to t_{i2} and that the point of intersection is the cell (e,e) . Q.E.D.

We shall now prove the following:

<u>Theorem 2. The totally symmetric loop $\mathcal{L}^*$ derived from any Kirkman-Steiner triple system contains a maximal identity-root.</u>

<u>Proof.</u> By lemma 2 every class in the given Kirkman-Steiner triple system determines an e-root of degree 2 in the Cayley table of $\mathcal{L}^*$, where e is the identity in $\mathcal{L}^*$. The method of exhibition in the lemma together with the fact that every pair of distinct elements in the triple system appears exactly once reveals that the transversal $t_{ik}(k=1,2)$ is orthogonal to $t'_{ik}(k=1,2)$ if $i \neq i'$ with cell (e,e) as the intersection point. Since there are (n-1)/2 classes, we have 2(n-1)/2 = n-1 pairwise orthogonal transversals sharing the cell (e,e), i.e., an identity-root of degree n-1 . Since the order of $\mathcal{L}^*$ is n+1, the proof is complete.

As an immediate application we have

<u>Corollary 1. Every totally symmetric loop of order n+1 derived from a Kirkman-Steiner triple system of order n implies the existence of a set consisting of at least a pair of mutually orthogonal Latin squares of order n .</u>

A proof of this corollary, together with some additional results, will be given in another paper. However, we should remark that, in particular, for n = 15, the corresponding pair of orthogonal Latin squares can be embedded in a set of three mutually orthogonal Latin squares of order 15, thus disproving MacNeish's [5] conjecture for order 15.

Before finishing, let us mention a few open problems.

(1) Prove or disprove that the totally symmetric loop of order n+1 derived from any arbitrary Steiner triple system of order n admits a maximal x-root.

(2) Characterize those loops whose Cayley tables admit a (1,1,...,1) orthogonal partition.

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