

~~Independent response residuals for testing~~

~~non-orthogonality~~
F-square and orthogonal F-square designs. A generalization of latin squares and orthogonal latin square designs.

~~SOME CONTRIBUTIONS TO THE THEORY OF F-SQUARES~~ (1)

by

A. Hedayat and E. Seiden (2)

Cornell University and Michigan State University

ABSTRACT

The theory of F-squares and especially of mutually orthogonal F-squares aims at a generalization of the concept of latin squares and widely considered search for mutually orthogonal latin squares. An F-square is an $n \times n$ square whose cells are filled with $k \leq n$ distinct elements in such a way that each element appears an equal number of times in each row and column of the square.

The concept of F-squares and mutually orthogonal F-squares is not entirely new. It has been considered directly or indirectly by D. J. Finney, W. T. Federer, G. H. Freeman, and S. Addelman. However, these authors considered the F-squares as a by-product of their general interest in experimental designs and were not concerned with the theory of F-squares per se. The first author of the present paper was inspired by the examples of F-squares brought forward by the previous authors. Having in mind the usefulness of the F-squares for research in the theory of the designs and its application in practical problems, he defined the concept of F-squares and mutually orthogonal F-squares and obtained some results concerning them. The purpose of this paper is to develop further the theory of F-squares and bring it to closer attention of mathematical statisticians. We plan to further the research in this area and present more results for publication shortly.

It may be worthwhile to point out at this stage that the theorems proved thus far aim at classifying some types of latin squares making use of their relation to F-squares. The concept of orthogonality of latin squares is generalized to F-squares and some of its implications to the problem of existence of orthogonal latin squares are investigated. It is pointed out here that using the concept of F-squares one can distinguish between two types of latin squares both mateless in respect to orthogonality but different in respect to their use in a broader sense of experimental designs.

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by

A. Hedayat and E. Seiden⁽²⁾

Cornell University and Michigan State University

0 - Introduction. The theory of F-squares and especially of mutually orthogonal F-squares aims at a generalization of the concept of latin squares and widely considered search for mutually orthogonal latin squares. An F-square is an $n \times n$ square whose cells are filled with $k \leq n$ distinct elements in such a way that each element appears an equal number of times in each row and column of the square.

The concept of F-squares and mutually orthogonal F-squares is not entirely new. It has been considered directly or indirectly by Finney [3,4,5], Federer [2], Freeman [6], and Addelman [1]. However, these authors considered the F-squares as a by-product of their general interest in experimental designs and were not concerned with the theory of F-squares per se. The first author of the present paper was inspired by the examples of F-squares brought forward by the previous authors. Having in mind the usefulness of the F-squares for research in the theory of the designs and its application in practical problems, he defined the concept of F-squares and mutually orthogonal F-squares and obtained some results concerning them [7]. The purpose of this paper is to develop further the theory of F-squares and bring it to closer attention of mathematical statisticians. We plan to further the research in this area and present more results for publication shortly.

It may be worthwhile to point out at this stage that the theorems proved thus far aim at classifying some types of latin squares making use of their relation to F-squares. The concept of orthogonality of latin squares is generalized to F-squares and some of its implications to the problem of existence of orthogonal latin squares are investigated. It is pointed out here that using the concept of F-squares one can distinguish between two types of latin squares both mateless in respect to orthogonality but different in respect to their use in a broader sense of experimental designs.

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I. F-Squares

Definition 1.1. Let $A = [a_{ij}]$ be an $n \times n$ matrix and let $\Sigma = (c_1, c_2, \dots, c_m)$ be the ordered set of distinct elements of A . In addition, suppose that for each $k=1, 2, \dots, m$, c_k appears precisely λ_k times ($\lambda_k \geq 1$) in each row and in each column of A . Then, A will be called a frequency square or, more concisely, an F-square on Σ of order n and frequency vector $(\lambda_1, \lambda_2, \dots, \lambda_m)$.

We now introduce some notation. A matrix A will be said to be an $F(n; \lambda_1, \lambda_2, \dots, \lambda_m)$ square if A is an F-square of order n and frequency vector $(\lambda_1, \lambda_2, \dots, \lambda_m)$. This notation may be contracted by the use of powers to denote successive equal values of λ 's. Thus $F(n; \lambda^m)$ represents $F(n; \lambda, \lambda, \dots, \lambda)$ while $F(n; \lambda_1^2, \lambda_3, \lambda_4^2, \lambda_6, \dots, \lambda_m)$ represents $F(n; \lambda_1, \lambda_1, \lambda_3, \lambda_4, \lambda_4, \lambda_6, \dots, \lambda_m)$. In particular, in an $F(n; \lambda^m)$ square, m is determined uniquely by n and λ ; hence we will represent such a square simply by $F(n; \lambda)$.

Examples:

1) Let $\Sigma = \{1, 2, 3\}$ then

1	2	3	3	2	1
2	3	1	1	3	2
3	1	2	2	1	3
3	1	2	2	1	3
2	3	1	1	3	2
1	2	3	3	2	1

is an $F(6; 2)$ square on Σ .

2) Let $\Sigma = \{1, 2, 3, 4\}$ then

1	2	3	4
2	1	4	3
3	4	1	2
4	3	2	1

is an $F(4;1)$ square on Σ , while

1	2	3	4	1
1	1	2	3	4
4	1	1	2	3
3	4	1	1	2
2	3	4	1	1

is an $F(5;2,1^3)$ on Σ .

Note that $F(n;1)$ square is simply a latin square of order n and exists for all n .

Lemma 1.1. An $F(n;\lambda_1,\lambda_2,\dots,\lambda_m)$ square on $\Sigma = \{a_1,a_2,\dots,a_m\}$ exists if and only if $n = \sum_{i=1}^m \lambda_i$.

Proof. The necessity of this condition is obvious by the definition of an F -square. Sufficiency can be proved as follows: Construct an $F(n;1)$ square on an ordered set $\Omega = (b_1,b_2,\dots,b_n)$. Partition Ω into m disjoint subsets S_1,S_2,\dots,S_m such that S_i contains λ_i elements. Define a many-one map σ from Ω onto Σ as follows:

$$\sigma(x) = a_i \text{ if and only if } x \in S_i, \quad i=1,2,\dots,m .$$

If we now apply σ to the elements of $F(n;1)$ square then we obtain an $F(n;\lambda_1,\lambda_2,\dots,\lambda_m)$ square on Σ .

Example .

Let us construct an $F(6;1,2,3)$ square on $\Sigma = \{A,B,C\}$. To do this, construct an $F(6;1)$ square on, say $\Omega = \{1,2,\dots,6\}$ such as

1 2 3 4 5 6
2 3 6 1 4 5
3 6 2 5 1 4
4 5 1 2 6 3
5 1 4 6 3 2
6 4 5 3 2 1 .

Then let $\Omega = S_1 \cup S_2 \cup S_3$ where $S_1 = \{1\}$, $S_2 = \{2,3\}$, and $S_3 = \{4,5,6\}$. And let $\sigma(S_1) \rightarrow A$, $\sigma(S_2) \rightarrow B$, and $\sigma(S_3) \rightarrow C$. Now applying σ to the above $F(6;1)$ square we obtain

A B B C C C
B B C A C C
B C B C A C
C C A B C B
C A C C B B
C C C B B A .

Before proceeding further, it should be noted that this idea is not entirely without practical importance. For example, if we let the elements of the m -set Σ to be treatments then an $F(n; \lambda_1, \lambda_2, \dots, \lambda_m)$ square on Σ is an experimental design having the properties that:

- a) Treatment effects \perp row effects (read treatment effects are orthogonal to row effects),
- b) Treatment effects \perp column effects, and if $\lambda_i = \lambda$, $i=1,2,\dots,m$, then we also have
- c) Treatment arrangement is balanced within rows and columns
- d) Row effects \perp column effects.

II. On the orthogonality of F-squares

Now we rigorously introduce the concept of the orthogonality of F-squares and then of latin squares as a special case of F-squares.

Definition 2.1. Given an F-square $F_1(n; \lambda_1, \lambda_2, \dots, \lambda_k)$ on a k-set $\Sigma = \{a_1, a_2, \dots, a_k\}$ and an F-square $F_2(n; u_1, u_2, \dots, u_t)$ on a t-set $\Omega = \{b_1, b_2, \dots, b_t\}$. Then we say F_2 is an orthogonal mate for F_1 (and write $F_2 \perp F_1$) if upon superposition of F_2 on F_1 , a_i appears $\lambda_i u_j$ times with b_j .

We now give a series of examples to elucidate the content of this definition. The sets Σ and Ω will for convenience be taken as the sets of integers $1, 2, \dots, k$ and $1, 2, \dots, t$ respectively.

	$F_1(6;1)$		$F_2(6;1,2,1^3)$
1)	1 2 3 4 5 6		1 2 3 4 2 5
	2 1 6 5 4 3		4 5 2 3 1 2
	3 4 1 2 6 5	⊥	2 3 2 5 4 1
	4 6 5 1 3 2		2 1 4 2 5 3
	5 3 2 6 1 4		5 4 1 2 3 2
	6 5 4 3 2 1		3 2 5 1 2 4

	$F_1(6;2)$		$F_2(6;2)$
2)	1 1 2 2 3 3		1 2 3 1 2 3
	1 2 1 3 2 3		2 3 1 1 2 3
	2 1 1 3 3 2	⊥	2 3 1 3 1 2
	2 3 3 1 1 2		3 1 2 2 3 1
	3 3 2 2 1 1		1 2 3 2 3 1
	3 2 3 1 2 1		3 1 2 3 1 2

	$F_1(5;1,2^2)$		$F_2(5;1^2,3)$
3)	1 2 2 3 3		1 2 3 3 3
	3 1 2 2 3		3 3 1 2 3
	3 3 1 2 2	⊥	2 3 3 3 1
	2 3 3 1 2		3 1 2 3 3
	2 2 3 3 1		3 3 3 1 2

	$F(4;1)$		$F(4;2)$
4)	1 2 3 4		1 1 2 2
	4 1 2 3		1 2 2 1
	3 4 1 2	⊥	2 2 1 1
	2 3 4 1		2 1 1 2

	$F(3;1)$		$F(3;1)$
5)	1 2 3		1 2 3
	2 3 1	⊥	3 1 2
	3 1 2		2 3 1

Definition 2.2. Let S_i be an n_i -set, $i=1,2,\dots,t$. Let F_i be an F-square of order n on the set S_i with frequency vector $\bar{\lambda}_i = (\lambda_{i1}, \lambda_{i2}, \dots, \lambda_{i, l_i})$. Then, we say $\{F_1, F_2, \dots, F_t\}$ is a set of t mutually (pair-wise orthogonal F-squares if $F_i \perp F_j$, $i \neq j$, $i, j=1, 2, \dots, t$. In particular, if $n_i=n$, $i=1, 2, \dots, t$, and every F_i is of type $F(n;1)$, i.e. a latin square of order n , then we denote such a set as an $O(n,t)$ set.

Example .

The following three F-squares are mutually orthogonal.

$F_1(5;2^2,1)$	$F_2(5;1^2,3)$	$F_3(5;1^3,2)$
1 2 3 1 2	1 2 3 3 3	1 2 3 4 4
2 1 2 3 1	3 3 1 2 3	3 4 4 1 2
1 2 1 2 3	2 3 3 3 1	4 1 2 3 4
3 1 2 1 2	3 1 2 3 3	2 3 4 4 1
2 3 1 2 1	3 3 3 1 2	4 4 1 2 3

Definition 2.3. Let Σ be an n -set. Let F be an F -square of order n on the set Σ with frequency vector $\bar{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_t)$. Then, we say F is of degree r with respect to the decomposition $n = u_1 + u_2 + \dots + u_s$ if there exist $r-1$ F -squares F_1, F_2, \dots, F_{r-1} on an s -set Ω with frequency vector $\bar{u} = (u_1, u_2, \dots, u_s)$ such that $\{F, F_1, \dots, F_{r-1}\}$ is a set of r mutually orthogonal F -squares and r is the largest such integer. In particular, if $\lambda_i = 1$; $u_j = 1$, $i=1, 2, \dots, t=n$, $j=1, 2, \dots, s=n$, i.e. $F, F_1, F_2, \dots, F_{r-1}$ are latin squares we say that F is of type $E(n, r)$. F is said to be orthogonally mateless with respect to the decomposition $n = u_1 + u_2 + \dots + u_s$ if its degree is one. F is said to be orthogonally rich if its degree is at least two with respect to every decomposition of n .

Definition 2.4. A set of r F -squares $\{F_1, F_2, \dots, F_r\}$ is said to be a mutually (pair-wise) orthogonally rich set if F_i is orthogonally rich and $F_i \perp F_j$, $i \neq j$, $i, j=1, 2, \dots, r$.

Now we give a series of examples to clarify the above definition.

Examples .

1. The degree of

$F =$	1 2 3 4 5 6
	2 3 6 1 4 5
	3 6 2 5 1 4
	4 1 5 2 6 3
	5 4 1 6 3 2
	6 5 4 3 2 1

is at least 2 with respect to the decomposition $6 = 1+1+1+1+2$. An orthogonal mate with respect to this decomposition is

$$F_1 = \begin{matrix} 1 & 2 & 5 & 5 & 3 & 4 \\ 5 & 1 & 3 & 2 & 4 & 5 \\ 3 & 5 & 4 & 1 & 5 & 2 \\ 5 & 4 & 2 & 3 & 1 & 5 \\ 4 & 3 & 5 & 5 & 2 & 1 \\ 2 & 5 & 1 & 4 & 5 & 3 \end{matrix} .$$

Indeed the degree of the above F-square is at least 2 with respect to any decomposition of 6 except $6 = 1+1+\dots+1$.

2. The following F-square

$$F = \begin{matrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \\ 3 & 4 & 1 & 2 \\ 2 & 3 & 4 & 1 \end{matrix}$$

is orthogonally mateless with respect to the decomposition $4 = 1+1+1+1$. However, its degree is at least 3 with respect to the decomposition $4 = 2 + 2$. A pair of mutually orthogonal mates for F with respect to this decomposition is:

$$F_1 = \begin{matrix} 1 & 1 & 2 & 2 \\ 1 & 2 & 2 & 1 \\ 2 & 2 & 1 & 1 \\ 2 & 1 & 1 & 2 \end{matrix} \quad \text{and} \quad \begin{matrix} 1 & 2 & 1 & 2 \\ 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & 1 \\ 2 & 1 & 2 & 1 \end{matrix} .$$

3. The following F-square is of degree 3 with respect to the decomposition $4 = 1+1+1+1$.

$$F = \begin{matrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{matrix} .$$

A pair of mutually orthogonal mates for F with respect to this decomposition is:

$$F_1 = \begin{matrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \\ 2 & 1 & 4 & 3 \end{matrix} \quad \text{and} \quad F_2 = \begin{matrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \end{matrix}$$

(By the following theorem, each of the F -squares in the last example above is orthogonally rich.)

The following theorem is of fundamental importance in the development of the theory of orthogonality of F -squares.

Theorem 2.1. An F -square R on an n -set Σ is orthogonally rich if and only if it has an orthogonal mate with respect to the decomposition $n = 1+1+\dots+1$.

Proof. To prove the sufficiency, let A be an orthogonal mate for R with respect to the decomposition $n = 1+1+\dots+1$. Then, by lemma 1.1 in section one, A can be used to generate any other F -square of order n with frequency vector $\bar{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_s)$ with $\lambda_1 + \lambda_2 + \dots + \lambda_s = n$. Since $A \perp R$, therefore those F -squares derived from A will be necessarily orthogonal to R . The necessity part follows directly from definition 2.3.

Examples .

$$1. \quad B = \begin{matrix} 1 & 2 & 5 & 5 & 3 & 4 \\ 5 & 1 & 3 & 2 & 4 & 5 \\ 3 & 5 & 4 & 1 & 5 & 2 \\ 5 & 4 & 2 & 3 & 1 & 5 \\ 4 & 3 & 5 & 5 & 2 & 1 \\ 2 & 5 & 1 & 4 & 5 & 3 \end{matrix}$$

is orthogonally rich. An $F(6;1)$ orthogonal mate for B is:

$$A = \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 6 & 1 & 4 & 5 \\ 3 & 6 & 2 & 5 & 1 & 4 \\ 4 & 1 & 5 & 2 & 6 & 3 \\ 5 & 4 & 1 & 6 & 3 & 2 \\ 6 & 5 & 4 & 3 & 2 & 1 \end{matrix} .$$

From theorem 2.1 we note that all other orthogonal mates for B corresponding to different decomposition of n , i.e. $F(6;6)$, $F(6;5,1)$, etc., can be obtained from A.

2.

$$B = \begin{matrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 2 & 5 & 4 \\ 2 & 4 & 5 & 1 & 3 \\ 4 & 5 & 1 & 3 & 2 \\ 5 & 3 & 4 & 2 & 1 \end{matrix}$$

is not orthogonally rich since it is mateless with respect to the decomposition $5 = 1+1+\dots+1$, Mann [9].

3.

$$B = \begin{matrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 1 & 2 \\ 2 & 3 & 4 & 5 & 1 \\ 4 & 5 & 1 & 2 & 3 \end{matrix}$$

is orthogonally rich. An $F(5;1)$ orthogonal mate for B is

$$A = \begin{matrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 1 & 2 & 3 \\ 5 & 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 1 & 2 \\ 2 & 3 & 4 & 5 & 1 \end{matrix} .$$

Note that $\{A,B\}$ is a mutually orthogonally rich set.

Definition 2.5. A sub-F-square of order t and frequency vector $(\lambda_1, \lambda_2, \dots, \lambda_k)$ denoted by $SFS(t; \lambda_1, \lambda_2, \dots, \lambda_k)$ is an F-square of order t and frequency vector $(\lambda_1, \lambda_2, \dots, \lambda_k)$ embedded in a larger F-square. If an F-square has only the trivial $SFS(1;1)$, then we say F contains no SFS.

Examples.

1. The underlined cells in the following F-square form an $SFS(2;1)$.

$$\begin{array}{cccc} \underline{1} & 1 & \underline{2} & 2 \\ 1 & 2 & 2 & 1 \\ 2 & 2 & 1 & 1 \\ \underline{2} & 1 & \underline{1} & 2 \end{array} .$$

2. The following F-square has no SFS of any order.

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \\ 3 & 4 & 5 & 1 & 2 \\ 4 & 5 & 1 & 2 & 3 \\ 5 & 1 & 2 & 3 & 4 \end{array} .$$

Proposition 2.1. If L is an $F(n;1)$ on an n -set Σ then: a) L cannot have an $SFS(t;1)$ if n is odd and $t \geq \frac{n+1}{2}$, b) L cannot have an $SFS(t;1)$ if n is even and $t \geq \frac{n}{2}+1$.

Proof. a) Suppose that L has an $SFS(t;1)$. We may assume without loss of generality that this $SFS(t;1)$ is formed by the first t rows and columns of L . Then the rectangle formed by the first t rows and the remaining $n-t$ columns could not include any element of the $SFS(t;1)$. On the other hand, $t \geq \frac{n+1}{2}$ implies $n-t < \frac{n-1}{2}$. Hence in each row some of the elements would have to appear more than once which contradicts the property of L . The proof of b) is analogous.

It is obvious by theorem 2.1 and the series of examples which were given before in this section that, if an F-square $F(n; \lambda_1, \lambda_2, \dots, \lambda_t)$ is orthogonally mateless with respect to the decomposition $n = 1+1+\dots+1$, then it is not necessarily mateless with respect to a coarser decomposition of n . However, if F is orthogonally mateless with respect to any then it is orthogonally mateless with respect to the decomposition $n = 1+1+\dots+1$.

Mann [9] proved that if a latin square L of order $n = 4t+2$ has a sub-latin square of order $2t+1$ then L is orthogonally mateless. In the language of the theory of F-squares Mann's result states that any $F(n;1)$, $n = 4t+2$, containing an $SFS(2t+1;1)$ is orthogonally mateless with respect to the decomposition $n = 1+1+\dots+1$. The following theorem provides us with a much stronger result, viz. it says that such an F-square is orthogonally mateless with respect to a coarser decomposition of n than $n = 1+1+\dots+1$. Therefore, Mann's [9] result turns out to be a special case of this theorem.

Theorem 2.2. Let L be an $F(n;1)$ square on an n -set Σ , $n = 4t+2$, t a positive integer. Then L is orthogonally mateless with respect to the decomposition $n = x+y$ if L contains an $SFS(2t+1;1)$ and x is an odd integer.

Proof. Let $m = 2t+1$. Denote the given $SFS(m;1)$ by L_1 . With no loss of generality we can assume that L_1 occupies the square formed by the first m rows and columns of L . Partition L as follows:

$$L = \begin{array}{c|c} L_1 & L_2 \\ \hline L_3 & L_4 \end{array} .$$

Note that $L_2, L_3,$ and L_4 are also SFS($m;1$). This is so because L is an $F(n;1)$ square. Note also that, the squares L_1 and L_4 (and similarly L_2 and L_3) contain the same elements of Σ . Now if L has an orthogonal mate with respect to the decomposition $n = x+y$, then this will mean that there is an $L' = F(n;x,y)$ on say $\Omega = \{A,B\}$ such that $L' \perp L$. This implies that upon the superposition of L' on L , A will appear x times on each row, column and element of Σ . Assume that r A 's appear on L_1 . This implies that r A 's will also appear on L_4 . Since the contents of L_1 and L_4 are the same, this means that, under the orthogonality assumption of L and L' , $x(2t+1)$ A 's should appear together on L_1 and L_4 or $2r = x(2t+1)$. Therefore, $r = x(2t+1)/2$. But if x is odd, $x(2t+1)/2$ has no integer solution, hence a contradiction.

As an immediate consequence to this theorem, we have:

Corollary 2.1. Any latin square of order $4t+2$ containing a sublatin square of order $2t+1$ is orthogonally mateless.

A very natural way of writing $F(n;1)$ squares are the cyclic ones. For this reason this family of F -squares has received a considerable amount of attention. For example, it is known that if L is an $F(n;1)$ square based on a cyclic permutation group of order n , then a) L is orthogonally rich if n is odd, b) L is orthogonally mateless with respect to any decomposition of n as long as "1" is in the decomposition (for instance see Hedayat and Federer [8]). While cyclic $F(n;1)$ squares are orthogonally mateless with respect to many decompositions of n if n is even, they are at least of degree 2 with respect to the decomposition $n = 2+2+\dots+2$ as the following theorem shows.

Theorem 2.3. If L is an $F(n;1)$ square and if L is based on a cyclic permutation group of order n (even) then L is at least of degree 2 with respect to the decomposition $n = 2+2+\dots+2$.

Proof. By construction. There is no loss of generality if we let G be the cyclic permutation group generated by $\begin{pmatrix} 1 & 2 & \dots & n \\ n & 1 & \dots & n-1 \end{pmatrix}$. If we now consider the entries of the cells on the main diagonal and the diagonals parallel to the main one we see that the entries of each diagonal together with its complement are occupied by the same elements. Considering each diagonal followed by its complement as an entity we shall construct presently a square L' orthogonal to L with respect to the decomposition $n = 2+2+\dots+2$. Filling in its n diagonals parallel to the main diagonal as follows: Take an ordered tuple of $n/2$ distinct elements say $\Omega = (a_1, a_2, \dots, a_{n/2})$. Fill in the n spaces of the main diagonal of L' repeating each element of Ω twice in the prescribed order. Permute cyclically the elements of Ω and fill in the diagonal starting with the second position in the first row as before. Continue the process until all the n diagonals are completed.

The following example elucidates the content of the above procedure.

$$L = \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 2 & 3 & 4 & 5 \\ 5 & 6 & 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 1 & 2 & 3 \\ 3 & 4 & 5 & 6 & 1 & 2 \\ 2 & 3 & 4 & 5 & 6 & 1 \end{matrix}$$

is an $F(6;1)$ square generated by $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 2 & 3 & 4 & 5 \end{pmatrix}$. An orthogonal mate for L with respect to the decomposition $6 = 2+2+2$ on the set $\Omega = \{A, B, C\}$ is

$$L' = \begin{matrix} A B C A B C \\ C A B C A B \\ C A B C A B \\ B C A B C A \\ B C A B C A \\ A B C A B C \end{matrix}$$

Corollary 2.2. Let n be an even integer. If L is an $F(n;1)$ square and if L is based on a cyclic permutation group G of order n , then L is at least of degree 2 with respect to any decomposition of n as long as every component of the decomposition is even.

Proposition 2.2. Let L be an $F(n;1)$ square based on a cyclic permutation group G of order n on an n -set Σ . Then L contains an $SFS(t;1)$ if and only if t divides n .

Proof. Clearly the first column of L forms a group isomorphic to G provided that we make each element of G correspond to the element of L into which it maps the unity of L . Obviously the subgroup of L corresponding to a subgroup of G of order t will form an $SFS(t;1)$ within the rows formed by the subgroup of G . The same will apply to the cosets of the subgroup of L .

Remarks. If $n = 4t+2$, t a positive integer, then by proposition 2.2 any $F(n;1)$ square based on a cyclic permutation group of order n has an $SFS(2t+1;1)$. Such an F -square by theorem 2.2 is orthogonally mateless with respect to the decomposition $n = x+(n-x)$ as long as x is odd, however, it is of degree 2 for those decompositions of n considered in Corollary 2.2.

The result of theorem 3.2 gives a temptation to conclude that a similar result might hold for the family of $F(n;1)$ squares, $n = 4t+3$, t a positive integer. Namely, if L is an $F(n;1)$ square, $n = 4t+3$, then L is orthogonally mateless with respect to the decomposition $n = (2t+1) + (2t+2)$ if L has an $SFS(2t+1;1)$. We discovered that this is not the case as the following example shows:

$$L = \begin{array}{ccc|ccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 1 & 6 & 7 & 4 & 5 \\ \hline 3 & 1 & 2 & 5 & 4 & 7 & 6 \\ 4 & 6 & 5 & 7 & 3 & 2 & 1 \\ 5 & 7 & 4 & 3 & 6 & 1 & 2 \\ 6 & 4 & 7 & 2 & 1 & 5 & 3 \\ 7 & 5 & 6 & 1 & 2 & 3 & 4 \end{array}$$

which is an $F(7;1)$ square with an $SFS(3;1)$. Note that L is not based on a cyclic group (see proposition 2.2). Indeed this square has an orthogonal mate with respect to the finest decomposition of 7 viz., $7 = 1+1+\dots+1$, and the following $F(;1)$ square is an example.

$$L' = \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 1 & 7 & 3 & 2 & 5 & 6 \\ 5 & 6 & 1 & 2 & 7 & 3 & 4 \\ 2 & 5 & 4 & 1 & 6 & 7 & 3 \\ 3 & 4 & 6 & 7 & 1 & 2 & 5 \\ 7 & 3 & 5 & 6 & 4 & 1 & 2 \\ 6 & 7 & 2 & 5 & 3 & 4 & 1 \end{array} .$$

Note however, that L' is based on a cyclic permutation group generated by

$$\left(\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 1 & 7 & 3 & 2 & 5 & 6 \end{array} \right) .$$

Definition 2.0. A directrix (transversal) of an $F(n; \lambda_1, \lambda_2, \dots, \lambda_t)$ square on a t -set $\Sigma = \{a_1, a_2, \dots, a_t\}$ is a collection of n cells such that the entries of these cells contain λ_i times a_i , and every row and column of F is represented in this collection.

Example. The underlined cells in the following $F(5; 2, 1, 2)$ square form a directrix.

$$\begin{array}{cccccc} \underline{1} & 5 & 1 & 4 & 5 & \\ 5 & \underline{4} & 5 & 1 & 1 & \\ 1 & 1 & 4 & 5 & \underline{2} & \\ 5 & 5 & 1 & \underline{1} & 4 & \\ 4 & 1 & \underline{2} & 5 & 1 & . \end{array}$$

Note that not every F -square has a directrix. For instance, the following $F(4; 1)$ square has no directrix.

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \\ 3 & 4 & 1 & 2 \\ 2 & 3 & 4 & 1 . \end{array}$$

The following results can easily be verified.

Proposition 2.3. Let L be an $F(n; 1)$ square, $n = 4t+3$, with an $SFS(2t+1; 1)$. Then L has a directrix if the $SFS(2t+1; 1)$ does.

Proposition 2.4. There does not exist a pair of orthogonal $F(n; 1)$ squares, $n = 4t+3$, having a pair of orthogonal $SFS(2t+1; 1)$.

Concluding remarks. We would like to emphasize that concepts akin to those expressed in part 2 are fundamental with broad application. For example, to determine whether or not a given $F(n;l)$ square, viz., a latin square of order n has an orthogonal mate, one can first, as a necessary condition, check whether L has an orthogonal mate with respect to a coarser decomposition than $n = 1+1+\dots+1$. Note that in general it is much easier to search for an orthogonal mate for L of order n with respect to a coarser decomposition of n than with respect to the finest one, viz., $n = 1+1+\dots+1$.

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