AUGMENTED FACTORIAL DESIGNS WITH ONE-WAY
ELIMINATION OF HETEROGENEITY\textsuperscript{1,2}

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SUMMARY

Some treatments could be augmented to each block of a factorial experiment so that the desired information on the additional levels can be obtained. Such augmented factorial designs are constructed in various ways. Also, a general approach to calculation of the expected values of the sum of squares for the designs is presented by using vector notation.

\textsuperscript{1} Parts of this paper are based on the thesis submitted by the author in partial fulfillment of the requirement of M.S. degree at Cornell University.

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I. Introduction

In the application of factorial patterns there sometimes arise situations where the experimenter feels that an additional level to each factor should be included in a standard factorial design. He could then augment some treatments to each block so that the desired information on the additional levels can be obtained without losing the information which would be obtained with the standard design. Such a design as the Plan (1) may be suitable to obtain additional information on main effects and certain interactions over more than the 2 levels in the $2^2$-factorial only by adding some additional treatments to each block.

Levels will be called standard level and augmented level. For example, in the Plan (1), the level symbols 0, 1 indicate standard levels and 2 indicates the augmented level. Plan (1) may be called an "Augmented $2^2$-factorial design", or "Incomplete block design with two different numbers of replicates for an $\frac{8}{9}$ fraction of a $3^2$-factorial experiment with block of 3 units". Augmented factorial designs such as Plan (1) are constructed in various ways. These are illustrated in the present paper.

Plan (1). Augmented $2^2$-factorial or an $\frac{8}{9}$ fraction of a $3^2$-factorial

<table>
<thead>
<tr>
<th>Rep. no.</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Confounding in standards</td>
<td>A</td>
<td>B</td>
</tr>
<tr>
<td>block no.</td>
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<td>12</td>
</tr>
<tr>
<td>standards</td>
<td>00</td>
<td>10</td>
</tr>
<tr>
<td></td>
<td>01</td>
<td>11</td>
</tr>
<tr>
<td>additional treatments</td>
<td>12</td>
<td>21</td>
</tr>
</tbody>
</table>
Federer [1961] presented the general approach for all augmented designs with one-way elimination of heterogeneity and Banerjee and Federer [1963, 1964] have given the methods of estimating of effects for any given fraction of a factorial experiment in which the treatments either occur zero or one time. Analysis for the augmented factorial designs are developed according to Federer's approach for the augmented designs and to Banerjee and Federer's theory for estimating the effects from the fractional replicate. Also, in the present paper, a general approach to calculation of the expected values of the sum of squares for designs with one-way elimination of heterogeneity will be done by using vector notation.

II. Construction

The construction of an augmented factorial design is illustrated with Plan (1). The Table 1 is obtained from the ordinary table of "the coefficients for single degree of freedom comparisons" in the $3^2$-factorial by rearranging the matrix to make non-singular partition matrices form the lower right corner of the table. Such matrices as can be made are indicated by bold lines in Table 1.

The full design matrix $X$, for instance Table 1, may be partitioned as

$$X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} X_{11} & X_{12} \\ \\ X_{21} & X_{22} \end{bmatrix} \end{bmatrix}$$

where the matrix $X_1$ is an $8 \times 9$ matrix and corresponds to the fractional replicate retained and $X_{11}$ is an $8 \times 8$ matrix. Banerjee and Federer [1963, 1964] showed that if the $q \times q$ matrix $X_{22}$ is non-singular then there exists
an \((n - q)/n\) fractional factorial. Plan (1) is obtained by such a procedure. Similarly, by using Table 1 some other augmented factorial designs can be constructed as indicated by the lines in the table. That is, the treatments below the line and corresponding parameters to the right of the line are omitted to obtain the fractional replicate.

Table 1. The coefficients for single degree of freedom comparisons in a \(3^2\)-factorial

<table>
<thead>
<tr>
<th>treatment</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
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<tbody>
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<td>-</td>
<td>+</td>
<td>-</td>
<td>+</td>
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</tr>
<tr>
<td>AL</td>
<td>+</td>
<td>-</td>
<td>+</td>
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<td>-2</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>-2</td>
</tr>
<tr>
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<td>+</td>
<td>0</td>
<td>-2</td>
<td>-</td>
<td>+</td>
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<td>0</td>
<td>2</td>
<td>-2</td>
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<td>0</td>
<td>-2</td>
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<tr>
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<td>-2</td>
<td>-2</td>
</tr>
<tr>
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<td>00</td>
<td>+</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>+</td>
</tr>
<tr>
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<td>+</td>
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<td>0</td>
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<td>-</td>
<td>+</td>
<td>0</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>11</td>
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<td>0</td>
<td>-2</td>
<td>0</td>
<td>-2</td>
<td>0</td>
<td>0</td>
<td>4</td>
</tr>
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<td>+</td>
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<td>+</td>
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<td>+</td>
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<tr>
<td>9</td>
<td>22</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td></td>
</tr>
</tbody>
</table>
Some plans for augmented factorials are given below. The parameters that can be estimated are listed below the plan.

Plan (2). Augmented $2^2$-factorial or a $\frac{6}{9}$ fraction of a $3^2$-factorial

<table>
<thead>
<tr>
<th>00</th>
<th>00</th>
<th>00</th>
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</thead>
<tbody>
<tr>
<td>01</td>
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<td>11</td>
</tr>
<tr>
<td>12</td>
<td>21</td>
<td></td>
</tr>
</tbody>
</table>

Standards in a randomized complete block design

Parameters: $M, A_L, A_Q, B_L, B_Q, A_LB_L$

Plan (3). Augmented $\frac{3}{4} 2^2$-factorial or a $\frac{6}{9}$ fraction of a $3^2$-factorial

(a) | block | 1 | 2 | 3 |
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
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<td>00</td>
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<td>00</td>
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<td></td>
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<tr>
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<tr>
<td>11</td>
<td>12</td>
<td>21</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Standards in a randomized complete block design

(b) | block | 1 | 2 | 3 |
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
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<td></td>
</tr>
<tr>
<td>11</td>
<td>12</td>
<td>21</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Standards in an incomplete block design with $v = 3, k = 2,$ and $b = 3$

Parameters are the same as in Plan (2)
The following plans are constructed by using rearranged tables of the coefficients for single degree of freedom comparisons in the $3^3$- or $4^2$-factorial.

Plan (4). Augmented $\frac{7}{8}$ $2^3$-factorial or a $\frac{14}{27}$ fractional replicate of a $3^3$-factorial

<table>
<thead>
<tr>
<th>block</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
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<th>6</th>
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<tbody>
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<td>100</td>
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<td>001</td>
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<td>010</td>
</tr>
<tr>
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<td>011</td>
<td>101</td>
<td>011</td>
<td>011</td>
<td>011</td>
</tr>
</tbody>
</table>

Standards in an incomplete block design with $v = 7$, $k = 3$, and $b = 7$


Plan (5). Augmented $\frac{5}{8}$ $2^3$-factorial or a $\frac{15}{27}$ fraction of a $3^3$-factorial

<table>
<thead>
<tr>
<th>block</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
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</tr>
</thead>
<tbody>
<tr>
<td>000</td>
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<td>000</td>
<td>001</td>
<td>000</td>
<td>100</td>
<td>010</td>
<td>000</td>
<td>100</td>
<td></td>
</tr>
<tr>
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<td>010</td>
<td>011</td>
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<td>110</td>
<td>111</td>
<td>012</td>
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<td>201</td>
<td>210</td>
<td>212</td>
<td>221</td>
<td></td>
</tr>
</tbody>
</table>

Standards in an incomplete block design with $v = 5$, $k = 2$, and $b = 10$

Plan (6). Augmented $2^9$-factorial or a $\frac{16}{27}$ fraction of a $3^3$-factorial

<table>
<thead>
<tr>
<th>block</th>
<th>11</th>
<th>12</th>
<th>21</th>
<th>22</th>
</tr>
</thead>
<tbody>
<tr>
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</tr>
<tr>
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<tr>
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<tr>
<td>111</td>
<td>110</td>
<td>111</td>
<td>110</td>
<td></td>
</tr>
<tr>
<td></td>
<td>012</td>
<td>122</td>
<td>210</td>
<td>221</td>
</tr>
<tr>
<td></td>
<td>021</td>
<td>201</td>
<td>212</td>
<td>121</td>
</tr>
</tbody>
</table>

Standards in a confounded factorial design

Parameters: $M, A_L, A_Q, B_L, B_Q, C_L, C_Q, A_L B_L, A_L C_L, B_L C_L, A_Q B_L,$

$A_Q C_L, A_Q B_Q, A_Q C_Q, A_Q B_L C_L$

Plan (7). Augmented $2^9$-factorial or a $\frac{20}{27}$ fraction of a $3^3$-factorial

<table>
<thead>
<tr>
<th>block</th>
<th>11</th>
<th>12</th>
<th>21</th>
<th>22</th>
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<td>000</td>
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</tr>
<tr>
<td>011</td>
<td>010</td>
<td>010</td>
<td>011</td>
<td>001</td>
<td>011</td>
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</tr>
<tr>
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<td>110</td>
<td>111</td>
<td>110</td>
<td>111</td>
<td>101</td>
<td></td>
</tr>
<tr>
<td></td>
<td>012</td>
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<td>122</td>
<td>210</td>
<td>212</td>
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<td>021</td>
<td>201</td>
<td>122</td>
<td>221</td>
<td>211</td>
</tr>
</tbody>
</table>

Standards in a confounded factorial design

Parameters: $M, A_L, A_Q, B_L, B_Q, C_L, C_Q,$

$A_L B_L, A_L C_L, B_L C_L, A_Q B_L, A_Q C_L, A_L B_Q,$

$B_Q C_L, A_Q C_Q, B_Q C_Q, A_Q B_Q, A_Q C_Q, B_Q C_Q,$

$A_L B_L C_L$
Plan (8). Augmented $2^3$-factorial or a $\frac{2^{27}}{27}$ fraction of a $3^3$-factorial

<table>
<thead>
<tr>
<th>BC</th>
<th>AC</th>
<th>AB</th>
<th>ABC</th>
</tr>
</thead>
<tbody>
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<td>000</td>
<td>001</td>
</tr>
<tr>
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</tbody>
</table>

Standards in a confounded factorial design

Plan (9). Augmented $3^2$-factorial or a $\frac{15}{16}$ fraction of a $4^2$-factorial

<table>
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<th>$AB^2$</th>
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</thead>
<tbody>
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<tr>
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<tr>
<td>12</td>
<td>10</td>
</tr>
<tr>
<td>21</td>
<td>22</td>
</tr>
</tbody>
</table>

Standards in a confounded factorial design
III. Intra-block Analysis

Let the yield of the $ijh^{th}$ observation be expressed by

$$Y_{ijh} = n_{ijh}(\mu + \tau_i + \rho_j + \beta_{jh} + \epsilon_{ijh}) \quad (3.1)$$

where $i = 1, \ldots, v =$ number of treatments; $j = 1, \ldots, r =$ number of complete blocks; $h = 1, \ldots, k_j =$ number of incomplete blocks in the $j^{th}$ complete block; $n_{ijh} = 1$ if the $i^{th}$ treatment occurs in the $h^{th}$ incomplete block of the $j^{th}$ complete block and zero otherwise. (Also $n_{ijh}$ could equal the number of times the $i^{th}$ treatment occurs in the $jh^{th}$ incomplete block but $n_{ijh}$ was restricted to 0 or 1 here.) It assumed that $\mu$ is the population mean, the effects are independent, $\tau_i$ is the treatment effect, $\rho_j$ is the complete block effect, $\beta_{jh}$ is the incomplete block effect and $\epsilon_{ijh}$ is a random effect with mean zero and variance $\sigma^2_\epsilon$.

The following equations are obtained after minimizing the residual sum of squares and after appropriate manipulation of the normal equations:

$$\hat{\mu} : n_{fg} \hat{\mu} - \sum_{j h} n_{j h} \hat{\mu}_j = Y_{..} - \sum_{j h} n_{j h} \overline{Y}_{..j h} = Q_{..} \quad (3.2)$$

$$\hat{\tau}_i + \hat{\beta}_{fg} : n_{fg} (\hat{\tau}_i + \hat{\beta}_{fg}) - \sum_{i} n_{i..} \sum_{j h} n_{ijh} (\hat{\tau}_i + \hat{\beta}_{jh}) = Y_{..} - \sum_{i} n_{ijh} \overline{Y}_{i..} = Q_{..} \quad (3.3)$$

Using matrix notation, equations (3.2) and (3.3) are expressed as follows:

$$C \hat{\mu} = Q_{..} \quad (3.4)$$

$$D(\hat{\tau} + \hat{\beta}) = Q_{..} \quad (3.5)$$
These equations are not independent, it being easily verified that the sums
of left-hand and of right-hand are both identically zero. To obtain a unique
solution, say from (3.2), we impose any condition, the simplest one (generally)
being \( \sum_{i} \hat{\xi} = 0 \). A convenient method of solving this set of equations is to
augment the equation (3.2) by introducing another unknown, say \( z \), and making
up the set of equations:

\[
\begin{bmatrix}
C & 1 \\
1' & 0
\end{bmatrix}
\begin{bmatrix}
\hat{\xi} \\
z
\end{bmatrix}
= 
\begin{bmatrix}
Q. \\
0
\end{bmatrix}
\tag{3.6}
\]

where \( l \) is a vector whose elements are unity.

Then, the solution of the equation is:

\[
\begin{bmatrix}
\hat{\xi} \\
z
\end{bmatrix}
= 
\begin{bmatrix}
C & 1 \\
1' & 0
\end{bmatrix}^{-1}
\begin{bmatrix}
Q. \\
0
\end{bmatrix}
= (c^{ie})
\begin{bmatrix}
Q. \\
0
\end{bmatrix}
\tag{3.7}
\]

As a result of the imposition of the chosen condition, any \( \hat{\xi} \) is the
estimate, in fact, of \( (\hat{\xi} - \bar{\xi}) \). This procedure is useful in that it produces
as a by-product the variances and covariances of the \( \hat{\xi}'s \). The variance of \( \hat{\xi} \)
is \( c^{ie} \sigma^2 \) and the covariance of \( \hat{\xi} \) and \( \hat{\xi} e \) is \( c^{ie} \sigma^2 \).

Let \( V = (c^{ie}) \) be the \( v \times v \) matrix obtained by deleting the last row and
the last column from the \( (v+1) \times (v+1) \) matrix \( (c^{ie}) \), and let \( L = (\lambda_{1e}) \) be the
\( m \times v \) matrix of contrast coefficients \( \lambda_{1e} \) such that \( \sum_{e=1}^{v} \lambda_{1e} = 0 \). Then, in
general, the variance-covariance matrix of \( L \cdot \hat{\xi} \) is expressed as

\[
\sigma^2_{\xi} L V L'
\tag{3.8}
\]

where \( \hat{\xi} \) is the \( v \times 1 \) column vector of treatment effects given by equation (3.7).

Suppose that a randomized layout in the Plan (1) is as given in Table 2.
Table 2. A randomized layout of Plan (1)

<table>
<thead>
<tr>
<th>block number</th>
</tr>
</thead>
<tbody>
<tr>
<td>11 12 21 22</td>
</tr>
<tr>
<td>01 10 02 01</td>
</tr>
<tr>
<td>12 11 00 20</td>
</tr>
<tr>
<td>00 21 10 11</td>
</tr>
</tbody>
</table>

In the experiment of Table 2, matrix C is:

\[
C = \frac{1}{3} \begin{bmatrix}
4 & -1 & -1 & 0 & -1 & 0 & -1 & 0 \\
-1 & 4 & 0 & -1 & -1 & 0 & 0 & -1 \\
-1 & 0 & 4 & -1 & 0 & -1 & -1 & 0 \\
0 & -1 & -1 & 4 & 0 & -1 & 0 & -1 \\
-1 & -1 & 0 & 0 & 2 & 0 & 0 & 0 \\
0 & 0 & -1 & -1 & 0 & 2 & 0 & 0 \\
-1 & 0 & -1 & 0 & 0 & 0 & 2 & 0 \\
0 & -1 & 0 & -1 & 0 & 0 & 0 & 2
\end{bmatrix}
\]

(3.9)

where the treatment order is: 00, 01, 10, 11, 12, 21, 02, 20

The coefficients associated with the variance-covariance matrix are

\[
v = (c^{ie}) : \frac{1}{32} \begin{bmatrix}
23 & -1 & -1 & -9 & 5 & -11 & 5 & -11 \\
-1 & 23 & -9 & -1 & 5 & -11 & -11 & 5 \\
-1 & -9 & 23 & -1 & -11 & 5 & 5 & -11 \\
-9 & -1 & -1 & 23 & -11 & 5 & -11 & 5 \\
5 & 5 & -11 & -11 & 47 & -17 & -9 & -9 \\
-11 & -11 & 5 & 5 & -17 & 47 & -9 & -9 \\
5 & -11 & 5 & -11 & -9 & -9 & 47 & -17 \\
-11 & 5 & -11 & 5 & -9 & -9 & -17 & 47
\end{bmatrix}
\]

(3.10)
Similarly, from equation (3.3), we can obtain the following solutions:

\[
\begin{bmatrix}
\hat{\beta}_1 + \hat{\beta}_{11} \\
\hat{\beta}_1 + \hat{\beta}_{12} \\
\hat{\beta}_2 + \hat{\beta}_{21} \\
\hat{\beta}_2 + \hat{\beta}_{22}
\end{bmatrix} = \frac{1}{4} \begin{bmatrix}
3 & -1 & 0 & 0 \\
-1 & 3 & 0 & 0 \\
0 & 0 & 3 & -1 \\
0 & 0 & -1 & 3
\end{bmatrix} \begin{bmatrix}
Q_{.11} \\
Q_{.12} \\
Q_{.21} \\
Q_{.22}
\end{bmatrix}
\]

(3.11)

and by setting up \( \hat{\beta}_{jh} = 0 \) for each \( jh \),

\[
\begin{bmatrix}
\hat{\rho}_1^* \\
\hat{\rho}_2^*
\end{bmatrix} = \frac{1}{4} \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} \begin{bmatrix}
Q_{.1} \\
Q_{.2}
\end{bmatrix}
\]

(3.12)

where \( Q_{.j} = Y_{.j} - \sum_{i=1}^{r} y_{ij} \bar{y}_{i..} \).

The sums of squares in the analysis of variance are computed as follows:

The sum of squares for treatment (eliminating all other effects) with \( (v-1) \) degrees of freedom is equal to \( \sum_{i=1}^{v} \bar{\pi}_i Q_{.i} \). The sum of squares for incomplete blocks within complete blocks (eliminating treatment effects) with \( (b-r) \) degrees of freedom is equal to \( \sum_{j=1}^{r} \sum_{h=1}^{r} (\hat{\beta}_{jh} + \hat{\beta}_{jh}) Q_{.jh} - \sum_{j=1}^{r} \sum_{h=1}^{r} \hat{\rho}_{j} Q_{.j} \). The residual sum of squares may be computed directly from the standards or as

\[
\text{Total} - \sum_{j=1}^{v} \sum_{h=1}^{r} k_{jh} Y_{jh}^2 / n_{jh} - \sum_{i=1}^{v} \bar{\pi}_{i} Q_{.i}
\]

\[
= \text{Total} - \sum_{j=1}^{r} \sum_{h=1}^{r} (\hat{\beta}_{jh} + \hat{\beta}_{jh}) Q_{.jh} - \sum_{i=1}^{v} Y_{i..}^2 / n_{i..} \text{ with } v-k-v+1 \text{ degrees of freedom, where } k = \sum_{j=1}^{r} k_j = b. \text{ This sum of squares divided by its degrees of freedom is an estimate of } \sigma^2.\]
IV. Factorial Analysis

This section is directly related to the paper by Banerjee and Federer [1963, 1964] on fractional replication. Let \( Y \) represent a column vector of \( n \) stochastic variates, \( y_1, y_2, \ldots, y_n \), let \( B \) represent a column vector of \( p \) unknown parameters, \( b_1, b_2, \ldots, b_p \) and let the known treatment design matrix be composed of \( n \) rows and \( p \) columns. Then the observational equations may be represented as

\[
Y = XB + e
\]  

where \( e \) is an \( n \times 1 \) column vector of error components, \( e_1, e_2, \ldots, e_n \) and where \( E(Y) = XB \). If \( n \geq p \), the least squares estimates of \( B \) are given by

\[
\hat{B} = [X'X]^{-1}X'Y
\]  

as in ordinary regression theory.

Suppose that we rewrite equation (4.1) as

\[
Y = \begin{bmatrix} Y_r \\ Y_0 \end{bmatrix} = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \begin{bmatrix} B_r \\ B_0 \end{bmatrix} + e
\]  

where \( X_{11} \) is \( (n-m) \times (p-m) \), \( X_{22} \) is \( m \times m \), \( Y_r \) and \( B_r \) are \( (n-m) \times 1 \) and \( (p-m) \times 1 \), \( B_0 \) and \( Y_0 \) are \( m \times 1 \). Banerjee and Federer [1963, 1964] showed that if \( X_{22} \) is non-singular then there exists a matrix \( \lambda \) such that

\[
\begin{bmatrix} X_{11} \\ X_{21} \end{bmatrix} = \begin{bmatrix} X_{11} \\ \lambda X_{11} \end{bmatrix}
\]  

and then

\[
\lambda = -X_{12}[X_{22}]^{-1}
\]
Since \[ \begin{bmatrix} X_{11} \\ \lambda' X_{11} \end{bmatrix} \] is diagonal in the factorial design, we can obtain \( \hat{\beta}_r \) by the least squares method as:

\[
\hat{\beta}_r = X_{11}'(I + \lambda \lambda') \gamma_r - X_{11}'(I + \lambda' \lambda) X_{12} \hat{\beta}_0
\]  \( (4.6) \)

Here, if we obtained the estimates of each treatment effect eliminating block and replicate effects, then any factorial effect (main effect or interaction) could be obtained using a linear combination of these estimates. We may write equations (4.1) and (4.2) as follows by using treatment effects.

\[
\hat{\beta} = X \hat{\beta}
\]  \( (4.7) \)

\[
\hat{\beta} = \begin{bmatrix} \hat{\beta}_r \\ \hat{\beta}_0 \end{bmatrix} = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \begin{bmatrix} \hat{\beta}_r \\ \hat{\beta}_0 \end{bmatrix}
\]  \( (4.8) \)

where \( \hat{\beta} \) represents a column vector of treatment effect estimators \( \hat{\beta}_1, \hat{\beta}_2, \ldots, \hat{\beta}_n \). Then, by the same procedure as (4.4) and (4.5), equation (4.6) becomes:

\[
\hat{\beta}_r = X_{11}'(I + \lambda \lambda') \hat{\beta}_r - X_{11}'(I + \lambda' \lambda) X_{12} \hat{\beta}_0
\]  \( (4.9) \)

Then

\[
\hat{\beta}_r + X_{11}'(I + \lambda \lambda') X_{12} \hat{\beta}_0 = X_{11}'(I + \lambda' \lambda) \hat{\beta}_r
\]  \( (4.9) \)

We could also have used \( \hat{\mu} + \hat{\beta}_i \) instead of \( \hat{\beta}_i \) and the parameter \( M \) would be estimated by \( \hat{\mu} \).

We defined \( X_{11}'(I + \lambda \lambda') \) as the matrix \( L \) of the contrast coefficients in Section III. In Plan (1)

\[
X_{22}^{-1} = 1
\]  \( (4.10) \)

Then, in the experiment of Table 2

\[
\lambda' = (-1, 2, 2, -4, 2, 2, -1, -1)
\]  \( (4.11) \)
where \( \hat{\mu} \) is an estimator of the \( \tau \), then \( E(\hat{\mu}) = 0 \) if \( E(\hat{\mu} + \hat{\tau}_1) = \mu \).

Using expression (3.7) and (3.10) in section III, we can express (4.12) as follows:

\[
\begin{bmatrix}
\hat{\mu} \\
\hat{A}_L \\
\hat{A}_Q \\
\hat{B}_L \\
\hat{B}_Q \\
\hat{A}_{LB} \\
\hat{A}_{QB} \\
\hat{A}_{QB} \\
\hat{A}_{LB}
\end{bmatrix} = \begin{bmatrix}
\frac{1}{3} (0 + + - + + 0 0) \\
\frac{1}{6} (-2 + 2 -4 2 3 -2 0) \\
\frac{1}{6} (0 + 0 -2 0 + 0 0) \\
\frac{1}{6} (-2 2 + -4 3 2 0 -2) \\
\frac{1}{6} (0 0 + -2 + 0 0 0) \\
\frac{1}{6} (0 + + -2 + + -) \\
\frac{1}{6} (-2 + -2 + 0 - 0) \\
\frac{1}{6} (- + 2 -2 0 + 0 -) \\
\frac{1}{6} (0 + -4 2 6 6 2 -2)
\end{bmatrix} \begin{bmatrix}
\hat{\tau}_{00} \\
\hat{\tau}_{01} \\
\hat{\tau}_{10} \\
\hat{\tau}_{11} \\
\hat{\tau}_{12} \\
\hat{\tau}_{21} \\
\hat{\tau}_{02} \\
\hat{\tau}_{20}
\end{bmatrix}
\]

In the expression (4.12), each variance of the parameter is obtained by using the expression (3.8) in section III.
V. Expected Value of the Sum of Squares

The generalized analysis of variance is developed below for all designs with one way elimination of heterogeneity. To be completely general, let the $i^{th}$ one of the $v$ treatments be replicated $n_i$ over the $b$ incomplete blocks of size $n_{.j}$. Let the yield of the $ijhu^{th}$ observation be expressed by

$$Y_{ijhu} = n_{ijhu}(\mu + \tau_i + \rho_j + \beta_{jh} + \epsilon_{ijhu}) \quad (5.1)$$

Here the subscripts $i$, $j$ and $h$ are already defined in section III, and also the property of $\mu$, $\tau$, $\rho$ and $\beta$ are assumed to be random independent variables with zero means and variances $\sigma^2_\tau$, $\sigma^2_\rho$ and $\sigma^2_\beta$ respectively, and $\epsilon$ are assumed to be the same. Let the subscript $u$ and multiplier $n_{ijhu}$ be defined as follows: $u = 1, \ldots, u_i = \text{number of times the } i^{th} \text{ treatment appears simultaneously in the same incomplete block, } n_{ijhu} = 1 \text{ if the } i^{th} \text{ treatment occurs } u^{th} \text{ time in the same block of the } h^{th} \text{ incomplete block of the } j^{th} \text{ complete block, and zero otherwise.}$

Let

$$Y_{ijhu} = Y_{ijhu} - n_{ijhu}u$$

Then we can write the equation (5.1) as follows, if the $i^{th}$ treatment is repeated at least $u$ times in the $jh^{th}$ block.

$$y_{ijhu} = 0\tau_1 + 0\tau_2 + \ldots + \tau_i + 0\tau_{i+1} + \ldots + 0\tau_v$$
$$+ 0\rho_1 + 0\rho_2 + \ldots + \rho_j + 0\rho_{j+1} + \ldots + 0\rho_r$$
$$+ 0\beta_{11} + 0\beta_{12} + \ldots + \beta_{jh} + 0\beta_{j+.h+1} + \ldots + 0\beta_{r_b} + \epsilon_{ijhu} \quad (5.2)$$

Using vector notation, let

$$Y = (g_1, g_2, \ldots, g_v) \cdot \tau + (h_{.1}, h_{.2}, \ldots, h_{.r}) \cdot \rho$$
$$+ (h_{11}, h_{12}, \ldots, h_{r_b}) \cdot \beta + \epsilon \quad (5.3)$$
Then

\[ g_i = \sum \sum \sum n_{ijhu} = n_i \ldots \]

\[ h_j = \sum \sum \sum n_{ijhu} = n_j \ldots \]

\[ h_{jh} = \sum \sum n_{ijhu} = \text{and } h_{jh} = n_{jh}. \]

\[ g_i \cdot g_e = 0 \text{ for } i \neq e \]

\[ h_j \cdot h_f = 0 \text{ for } j \neq f \]

\[ h_{jh} \cdot h_{fg} = 0 \text{ and } h_{jh} \cdot h_{fg} = 0 \text{ for } j \neq f \text{ and } h \neq g \]

\[ g_i \cdot h_j = n_{j} \ldots \]

\[ g_i \cdot h_{jh} = \sum n_{ijhu} = n_{ijh}. \]

\[ \sum g_i = \sum h_j = \sum \sum h_{jh} = 1 \text{ = a vector whose elements are unity} \]

\[ \text{and} \]

\[ g_i \cdot y = \sum \sum \sum y_{ijhu} = y_i \ldots \]

\[ h_j \cdot y = \sum \sum \sum y_{ijhu} = y_j \ldots \]

\[ h_{jh} \cdot y = \sum \sum y_{ijhu} = y_{jh}. \]

From the least squares method, the following equations are obtained:

\[ g_i \cdot y = (0, 0, \ldots, g_i^2, 0, \ldots, 0) \hat{\beta} + (g_i \cdot h_1, \ldots, g_i \cdot \hat{h}_2, \ldots, \hat{g}_i \cdot h_{rk}) \cdot \hat{\beta} \]

\[ (5.4) \]
\[ h_j \cdot y = (h_j \cdot g_1, h_j \cdot g_2, \ldots, h_j \cdot g_v) \cdot \hat{r} + (0, 0, \ldots), \]
\[ h_{j}^{0}, 0, \ldots, 0) \cdot \hat{o} + (0, 0, \ldots, h_{j} \cdot h_{j1}, h_{j} \cdot h_{j2}, \ldots, \]
\[ h_{j} \cdot h_{jk_1}, 0, \ldots, 0) \hat{o} \quad (5.5) \]

\[ h_{jh} \cdot y = (h_{jh} \cdot g_1, h_{jh} \cdot g_2, \ldots, h_{jh} \cdot g_v) \cdot \hat{r} + (0, 0, \ldots, \]
\[ h_{jh}^{0}, 0, \ldots, 0) \cdot \hat{o} + (0, 0, \ldots, h_{jh}^{0}, 0, \ldots, 0) \hat{o} \quad (5.6) \]

Let
\[ c_1 = g_1 - \sum \sum (g_1 \cdot h_{jh}) h_{jh} / h_{jh}^{0} \quad (5.7) \]

Then
\[ c_1^{0} = n_{i} \ldots - \sum \sum (n_{i,jh.})^2 / n_{j,h} = c_1 \cdot g_1 \quad (5.8) \]

\[ c_1 \cdot c_e = - \sum \sum (n_{i,jh.})(n_{e,jh.}) / n_{j,h} = c_1 \cdot g_e \text{ for } i \neq e \quad (5.9) \]

and
\[ \sum_{e} c_1 \cdot c_e = c_1 \cdot \sum_{e} g_e = c_1 \cdot l = n_{i} \ldots - \sum \sum (n_{i,jh.}) = 0 \quad (5.10) \]

From (5.4), (5.6)
\[ g_1 \cdot y - \sum \sum (g_1 \cdot h_{jh}) / h_{jh}^{0} (h_{jh} \cdot y) \]
\[ = (- \sum \sum (g_1 \cdot h_{jh})(g_1 \cdot h_{jh}) / h_{jh}^{2}, \ldots, \quad (5.11) \]
\[ g_1^{2} - \sum \sum (g_1 \cdot h_{jh})^{2} / h_{jh}^{2}, \ldots, \]
\[ - \sum \sum (g_1 \cdot h_{jh})(g_v \cdot h_{jh}) / h_{jh}^{2} \hat{r} \]
We can rewrite (5.11) as follows:

\[ c_1 \cdot \bar{Y} = (c_1 \cdot \bar{g}_1, c_1 \cdot \bar{g}_2, \ldots, c_1 \cdot \bar{g}_i, \ldots, c_1 \cdot \bar{g}_v) \cdot \hat{\beta} \]

\[ = (c_1 \cdot \bar{c}_1, c_1 \cdot \bar{c}_2, \ldots, c_1 \cdot \bar{c}_i, \ldots, c_1 \cdot \bar{c}_v) \cdot \hat{\beta} \]

(5.12)

and

\[ c_1 \cdot \bar{Y} = c_1 \cdot (\bar{Y} - \mu) = c_1 \cdot \bar{Y} \]

\[ = Y_1 \ldots - \sum \sum \sum n_{ijhu} \bar{Y}_{jh} = Q_1 \ldots \]

(5.13)

Then, we can obtain the following solution:

\[
\begin{bmatrix}
\hat{\beta} \\
0
\end{bmatrix} = \begin{bmatrix}
(c_1 \cdot c_e) & 1 \\
1' & 0
\end{bmatrix}^{-1} \begin{bmatrix}
Q_r \\
0
\end{bmatrix}
\]

(5.14)

Let (5.14) be expressed as follows:

\[
\begin{bmatrix}
\hat{\beta} \\
0
\end{bmatrix} = \begin{bmatrix}
c & 1 \\
1' & 0
\end{bmatrix}^{-1} \begin{bmatrix}
Q_r \\
0
\end{bmatrix}
\]

(5.15)

\[
= \begin{bmatrix}
(e^{ie}) & Y \\
Y' & 0
\end{bmatrix} \begin{bmatrix}
Q_r \\
0
\end{bmatrix}
\]

(5.16)

\[
= \begin{bmatrix}
V & Y \\
Y' & 0
\end{bmatrix} \begin{bmatrix}
Q_r \\
0
\end{bmatrix}
\]

(5.17)

where \( \sum_{i=1}^{V} \gamma_i = 1 \)

The sum of squares of among treatments (eliminating complete and incomplete block effects) is:

\[ \sum_{i} \hat{\beta}_i Q_{1i} \ldots \]

(5.18)
Then

\[ E[\hat{\theta} \cdot \hat{\theta}] = E\left[ \sum_{i} \hat{\theta}_i \cdot \mathbf{y}_i \right] = E\left[ \sum_{i,E} \left( \sum_{e} \hat{\theta}_e \cdot \mathbf{y}_e \right) \left( \hat{\theta}_i \cdot \mathbf{y}_i \right) \right] = \text{tr} \, V \, CC' \sigma^2 + \text{tr} \, V \, \Sigma^2 (5.19) \]

While

\[ \text{tr} \, V \, CC' = \sum_{i} \left( \mathbf{c}_i \cdot \mathbf{c}_i - \gamma_i \sum_{e} \mathbf{c}_i \cdot \mathbf{c}_e \right) = \sum_{i} \left( \mathbf{c}_i \cdot \mathbf{c}_i \right) = \text{tr} \, C (5.20) \]

Because from (5.10) \( \sum_{e} \mathbf{c}_i \cdot \mathbf{c}_e = 0 \)

\[ \text{tr} \, V \, C = \sum_{i=1}^{v} (1 - \gamma_i) = v - 1. (5.21) \]

Then \( E\left[ \sum_{i} \hat{\theta}_i \cdot \mathbf{y}_i \right] = (\text{tr} \, C) \sigma^2 + (v - 1) \Sigma^2 (5.22) \)

Let

\[ \frac{d_jh}{h_jh} = \sum_{i} \left( g_i \cdot h_{ij} \right) g_i / g_i^2 (5.23) \]

Then \( \frac{d_jh}{d_jh} = n \cdot \frac{1}{\sum_{i} (n_{ijh})^2 / n_i} = \frac{d_jh}{h_{ijh}} (5.24) \)

\[ \frac{d_jh}{d_jh} = - \sum_{i} (n_{ijh}) (n_{fg}) / n_i = \frac{d_jh}{h_{fg}} \text{ for } gh \neq fg (5.25) \]

and \( \sum_{f,g} \frac{d_jh}{d_jh} \cdot \frac{d_jh}{d_jh} = 0 (5.26) \)

From (5.4) and (5.6)

\[ \frac{d_jh}{d_jh} \cdot \mathbf{y} = \left( \frac{d_jh}{d_{j1}}, \frac{d_jh}{d_{j2}}, \ldots, \frac{d_jh}{d_{jk}} \right) (\hat{\theta} + \hat{\theta}) \]

\[ = \left( \frac{d_jh}{d_{j1}}, \frac{d_jh}{d_{j2}}, \ldots, \frac{d_jh}{d_{jk}} \right) (\hat{\theta} + \hat{\theta}) (5.27) \]
and \( d_{jh} \cdot \bar{y} = d_{jh} \cdot (\bar{y} - \mu) = d_{jh} \cdot \bar{y} \)

\[
y_{jh} - \sum_{i=1}^{n_{ijh}} \bar{y}_{i} = Q_{jh}. \tag{5.28}
\]

Then, we can obtain the following solution:

\[
\begin{bmatrix}
\hat{\rho} + \hat{\beta} \\
0
\end{bmatrix} = \begin{bmatrix}
d_{jh} \cdot d_{fg} & \frac{1}{2} \\
1 & 0
\end{bmatrix}^{-1} \begin{bmatrix}
Q_{\rho \beta} \\
0
\end{bmatrix} \tag{5.29}
\]

Let (5.29) be expressed as follows:

\[
\begin{bmatrix}
\hat{\rho} + \hat{\beta} \\
0
\end{bmatrix} = \begin{bmatrix}
D & \frac{1}{2} \\
1' & 0
\end{bmatrix} \begin{bmatrix}
Q_{\rho \beta} \\
0
\end{bmatrix} \tag{5.30}
\]

\[
= \begin{bmatrix}
d_{jhfg} & \hat{\beta} \\
\hat{\beta}' & 0
\end{bmatrix} \begin{bmatrix}
Q_{\rho \beta} \\
0
\end{bmatrix} \tag{5.31}
\]

\[
= \begin{bmatrix}
B & \hat{\delta} \\
\hat{\delta}' & 0
\end{bmatrix} \begin{bmatrix}
Q_{\rho \beta} \\
0
\end{bmatrix} \tag{5.32}
\]

where \( \sum_{j=1}^{r} \sum_{h=1}^{k} \hat{\delta}_{jh} = 1 \)

The sum of squares among complete and incomplete blocks (eliminating treatment effects) is:

\[
\sum \sum (\hat{\rho}_{j} + \hat{\beta}_{jh})Q_{jh}. \tag{5.33}
\]

Then \( E[\sum \sum (\hat{\rho}_{j} + \hat{\beta}_{jh})Q_{jh}.] = E[\sum \sum \sum \sum d_{jhfg}(d_{fg} \cdot \bar{y})(d_{jh} \cdot \bar{y})] \)
\[
\sum \sum \sum d_{jhf} g \left[ \sum \sum (d_{fg} \cdot d_{uv}) (d_{uf} \cdot d_{jg}) \sigma^2 + \sum \sum (d_{fg} \cdot d_{uv})(d_{uv} \cdot d_{jg}) \sigma^2 \right] + (d_{fg} \cdot d_{jg}) \sigma^2 \]  
(5.34)

while
\[
\sum \sum \sum \sum d_{jhf} g \left( \sum \sum (d_{fg} \cdot d_{uv})(d_{uv} \cdot d_{jg}) \right) = \text{tr} BDD = \text{tr} D  
(5.35)
\]

and
\[
\sum \sum \sum \sum d_{jhf} g (d_{fg} \cdot d_{jg}) = \text{tr} BD = (b - 1)  
(5.36)
\]

and
\[
\sum \sum \sum \sum d_{jhf} g \left( \sum (d_{fg} \cdot d_{u} \cdot d_{u} \cdot d_{jg}) \right) = \sum \sum \sum \sum (d_{jg} \cdot d_{jg})  
(5.37)
\]

Then we can write (5.34) as follows:
\[
E \left[ \sum \sum (\hat{\rho}_j + \hat{\rho}_{jg}) \cdot d_{jg} \right]  
= \sum \sum \sum (d_{jg} \cdot d_{jg}) \sigma^2 + (\text{tr} D) \sigma^2 + (b - 1) \sigma^2  
(5.38)
\]

If the same treatments do not appear in another incomplete block within the same complete block, even when the same treatment appears more than one time within the same incomplete block, then
\[
d_{jg} \cdot d_{jg} = 0 \quad \text{for } h \neq g
\]

\[
\sum \sum \sum (d_{jg} \cdot d_{jg}) = \sum \sum (d_{jg} \cdot d_{jg})  
(5.39)
\]

Then, we can write (5.38) as follows:
Let
\[ E\left[ \sum_{j} \sum_{h} (\hat{\beta}_{j} + \hat{\beta}_{jh})(\hat{d}_{jh} \cdot \mathbf{y}) \right] \]
\[ = (\text{tr } D)\sigma^2 + (\text{tr } D)\rho^2 + (b - 1)\sigma^2 \]  
\hspace{1cm} (5.40)

Let
\[ \hat{d}_{j} = \mathbf{h}_{j} - \sum_{i} (\mathbf{g}_{i} \cdot \mathbf{h}_{j})\mathbf{g}_{i}/\sigma^2_{i} \]  
\hspace{1cm} (5.41)

Then \[ \hat{d}_{j} = \sum_{h} \hat{d}_{jh} \]  
\hspace{1cm} (5.42)

\[ \sigma_{j}^2 = \mathbf{n}_{j} \cdot \mathbf{n}_{j} - \sum_{i} \frac{(\mathbf{n}_{i,j})^2}{\mathbf{n}_{i}} = \hat{d}_{j} \cdot \mathbf{h}_{j} \]  
\hspace{1cm} (5.43)

\[ \hat{d}_{j} \cdot \hat{d}_{f} = - \sum_{i} \frac{(\mathbf{n}_{i,j})(\mathbf{n}_{i,f})}{\mathbf{n}_{i}} = \hat{d}_{j} \cdot \mathbf{h}_{f}, \text{ for } j \neq f \]  
\hspace{1cm} (5.44)

and \[ \sum_{f} \hat{d}_{j} \cdot \hat{d}_{f} = 0 \]  
\hspace{1cm} (5.45)

Set \( \hat{\beta} = 0 \), then from (5.4) and (5.5)
\[ \hat{d}_{j} \cdot \mathbf{y} = (\hat{d}_{j} \cdot \mathbf{h}_{1}, \hat{d}_{j} \cdot \mathbf{h}_{2}, \ldots, \hat{d}_{j} \cdot \mathbf{h}_{r}) \cdot \mathbf{g}^{*} \]
\[ = (\hat{d}_{j} \cdot \mathbf{d}_{l}, \hat{d}_{j} \cdot \mathbf{d}_{2}, \ldots, \hat{d}_{j} \cdot \mathbf{d}_{r}) \cdot \mathbf{g}^{*} \]  
\hspace{1cm} (5.46)

and \[ \hat{d}_{j} \cdot \mathbf{y} = \hat{d}_{j} \cdot (\mathbf{y} - \mathbf{\mu}) = \hat{d}_{j} \cdot \mathbf{y} \]
\[ = \mathbf{y}_{j} - \sum_{i} \sum_{h} \sum_{u} \mathbf{y}_{i} = q_{j} \]  
\hspace{1cm} (5.47)

Then, we can obtain the following solution:
\[ \left[ \mathbf{g}^{*} \right] = \left[ \left[ \hat{d}_{j} \cdot \mathbf{d}_{f} \right] 1 \right]^{-1} \left[ \mathbf{q}_{\rho} \right] \]  
\hspace{1cm} (5.48)
Let (5.48) be expressed as follows:

\[
\begin{bmatrix}
\varrho^* \\
0
\end{bmatrix} = \begin{bmatrix}
P & 1 \\
1' & 0
\end{bmatrix}^{-1} \begin{bmatrix}
\varrho_p \\
0
\end{bmatrix} \tag{5.49}
\]

\[
= \begin{bmatrix}
[a_j] & \delta_j \\
\delta_j' & 0
\end{bmatrix} \begin{bmatrix}
\varrho_p \\
0
\end{bmatrix} \tag{5.50}
\]

\[
= \begin{bmatrix}
G & \delta_j \\
\delta_j' & 0
\end{bmatrix} \begin{bmatrix}
\varrho_p \\
0
\end{bmatrix} \tag{5.51}
\]

where \( \sum_j \delta_j = 1 \)

and let

\[
p^* = \begin{bmatrix}
d_1 \cdot d_{11} & d_1 \cdot d_{12} & \cdots & d_1 \cdot d_r & d_r \cdot d_{11} & d_r \cdot d_{12} & \cdots & d_r \cdot d_{r_k}
\end{bmatrix} \tag{5.52}
\]

The sum of squares among complete blocks (eliminating treatments, ignoring incomplete blocks) is:

\[
\sum_j \rho^*_{ij} \cdot Q_{ij} \tag{5.53}
\]

Then \( E[\sum_i \rho^*_{ij} \cdot Q_{ij}] \)
\[
E \left[ \sum_{j} \sum_{f} \left( \sum_{d_{ef}} \cdot \sigma + \sum_{d_{eq}} \cdot \beta + \varepsilon \right) \right] = \left( \text{tr } P \right) \sigma^2 + \left( \text{tr } \text{GD}^* \text{D}^* \right) \sigma^2 + (r - 1) \varepsilon^2 \quad (5.54)
\]

Here
\[
\text{tr } P = \sum_{j} d_{jh} \cdot d_{jh}.
\]

Then, by expressions (5.38), (5.54) and (5.55), the expected value of the sum of squares among incomplete blocks (eliminating treatments and complete blocks) is:
\[
E \left[ \sum_{j} \sum_{h} \left( \hat{\beta}_{jh} + \hat{\beta}_{j} \right) Q_{jh} \cdot \sum_{j} \hat{\beta}_{j} Q_{j} \right] = \left( \text{tr } D - \text{tr } \text{GD}^* \text{D}^* \right) \sigma^2 + (b - r) \varepsilon^2 \quad (5.56)
\]

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VI. References

