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Kirkman-Steiner Triple Systems and Sets of Mutually Orthogonal Latin Squares

BU-277-M

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Abstract

It is shown that every Kirkman-Steiner triple system of order $v \equiv 3 \pmod 6$ implies the existence of a set consisting of at least one pair of mutually orthogonal latin squares of order v. The combinatorial structure of this set is different from those of known sets of orthogonal latin squares in the literature and this might prove to be useful for the construction of other designs and combinatorial systems derivable from sets of mutually orthogonal latin squares. The case $v = 15$ leads to a new result, namely the existence of a set consisting of three mutually orthogonal latin squares of order 15.

Preparatory Definitions

1. Let Σ be a v-set, $v \equiv 1,3 \pmod 6$. Then a Steiner triple system of order v on Σ is a collection of $v(v-1)/6$ unordered triplets (x,y,z) with x,y,z in Σ, such that every pair of distinct elements of Σ belongs to exactly one triple. A triple system of order $v \equiv 3 \pmod 6$ is said to be a Kirkman-Steiner triple system of order n if it is a Steiner triple system with the following additional stipulation: The set of triples can be partitioned into $r = (v-1)/2$ distinct classes such that the totality of elements in each class exhaust the set on which the

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Presented at the 125^{th} meeting of the Institute of Mathematical Statistics held at the University of North Carolina, Chapel Hill, during the period May 5-7, 1970.

system is defined. In experiments design jargon, a Steiner triple system (Kirkman-Steiner triple system) of order v is called a balanced incomplete block design (resolvable balanced incomplete block design) with parameters $b = v(v-1)/6$, v, $r = (v-1)/2$, $k = 3$ and $\lambda = 1$.

2. Let Ω be an n-set. Then L is a latin square of order n on Ω if L is an $n \times n$ matrix with the property that each row and column of L is an n-permutation of elements of Ω. A collection of n cells in L is said to form a transversal (directrix) for L if the entries of these cells exhaust the set Ω and every row and column of L is represented in this collection. Two transversals are said to be parallel if they have no cell in common. Let L_1 and L_2 be two latin squares of order n on the n-set $\Omega_1 = \{a_1, a_2, \cdots, a_n\}$ and $\Omega_2 = \{b_1, b_2, \cdots, b_n\}$ respectively. Then we say L_2 is an orthogonal mate for L_1 if, upon superposition of L_2 on L_1, a_i in L_1 appears with b_j in L_2 for all $i,j=1,2,\cdots,n$.

<u>Preparatory Lemma, Proposition, Throrem</u>

<u>Lemma</u>. <u>If</u> L <u>is a latin square of order</u> v, <u>then</u> L <u>can have an orthogonal mate if and only if it has</u> v-1 <u>parallel transversals</u>.

<u>Proposition</u>. <u>If</u> L <u>is a latin square of order</u> n, <u>then</u> (a) L <u>cannot have a sub-latin square of order</u> t <u>if</u> n <u>is odd and</u> $t \geq (n+1)/2$, (b) L <u>cannot have a sub-latin square of order</u> t <u>if</u> n <u>is even and</u> $t \geq n/2 + 1$.

<u>Theorem</u>. <u>There exists a Kirkman-Steiner triple system of order</u> v <u>for all</u> $v \equiv 3 \pmod 6$, [7].

Results

Theorem. *Every Kirkman-Steiner triple system of order* v *implies the existence of a set consisting of at least a pair of mutually orthogonal latin squares of order* v.

Proposition. *There exists a set consisting of at least three mutually orthogonal latin squares of order* 15, [3].

Example. We consider v = 15, for it proved to be an interesting case as we shall see.

The following collection of triples is a Kirkman-Steiner triple system of order 15 on $\Sigma = \{A,B,\cdots,O\}$.

{E,F,G}	{E,C,I}	{E,L,O}	{E,H,D}	{E,J,N}	{E,A,M}	{E,K,B}
{C,M,O}	{F,J,O}	{F,I,D}	{F,A,K}	{F,B,M}	{F,L,N}	{F,C,H}
{L,H,K}	{L,D,B}	{C,N,K}	{C,J,B}	{C,L,A}	{C,G,D}	{L,G,J}
{A,J,D}	{G,K,M}	{G,A,B}	{L,I,M}	{G,H,I}	{H,B,O}	{A,I,O}
{I,N,B}	{H,A,N}	{H,J,M}	{G,N,O}	{D,K,O}	{J,I,K}	{N,D,M}

The corresponding pair of orthogonal latin squares of order 15 associated with this system of triples are

A	G	L	J	M	K	B	N	O	D	F	C	E	H	I
G	B	J	L	K	M	A	O	N	C	E	D	F	I	H
L	J	C	G	I	H	D	F	E	B	N	A	O	K	M
J	L	G	D	H	I	C	E	F	A	O	B	N	M	K
M	K	I	H	E	G	F	D	C	N	B	O	A	J	L
K	M	H	I	G	F	E	C	D	O	A	N	B	L	J
B	A	D	C	F	E	G	I	H	L	M	J	K	O	N
N	O	F	E	D	C	I	H	G	M	L	K	J	A	B
O	N	E	F	C	D	H	G	I	K	J	M	L	B	A
D	C	B	A	N	O	L	M	K	J	I	G	H	E	F
F	E	N	O	B	A	M	L	J	I	K	H	G	C	D
C	D	A	B	O	N	J	K	M	G	H	L	I	F	E
E	F	O	N	A	B	K	J	L	H	G	I	M	D	C
H	I	K	M	J	L	O	A	B	E	C	F	D	N	G
I	H	M	K	L	J	N	B	A	F	D	E	C	G	O

O	C	E	H	M	K	J	I	G	A	D	L	F	B	N
J	O	D	I	G	L	C	M	A	K	N	B	E	H	F
L	K	O	M	I	N	F	G	B	D	J	E	A	C	H
A	B	F	O	D	C	M	K	J	H	E	I	G	N	L
F	N	B	K	O	A	H	D	I	E	G	C	M	L	J
D	E	G	J	H	O	A	N	C	B	K	F	L	M	I
C	J	M	F	A	H	O	E	L	G	B	N	I	D	K
B	F	N	D	K	G	L	O	E	C	A	H	J	I	M
N	H	I	C	B	J	E	L	O	M	F	K	D	A	G
H	D	K	A	L	I	N	J	F	O	M	G	C	E	B
K	G	C	L	N	D	I	H	M	F	O	A	B	J	E
E	I	L	B	J	M	G	A	D	N	H	O	K	F	C
M	L	H	N	F	E	B	C	K	J	I	D	O	G	A
I	A	J	G	E	F	K	B	H	L	C	M	N	O	D
G	M	A	E	C	B	D	F	N	I	L	J	H	K	O

The above left-hand side latin square also admits the following orthogonal mate which has a special feature.

E	A	K	G	N	D	M	O	C	F	H	B	J	L	I
N	E	H	M	L	G	J	I	K	D	O	A	F	B	C
G	D	E	I	M	O	C	L	A	H	N	F	K	J	B
K	C	J	E	H	A	N	D	M	B	F	I	L	O	G
A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
H	K	F	N	O	E	L	G	J	A	D	C	B	I	M
C	I	M	A	K	B	E	F	G	L	J	O	N	H	D
F	J	B	H	D	K	O	E	L	C	A	M	I	G	N
M	G	N	C	J	I	K	B	E	O	L	D	H	F	A
O	H	D	L	I	N	B	M	F	E	G	K	A	C	J
D	F	A	O	G	H	I	N	B	K	E	J	C	M	L
L	N	O	J	B	M	F	C	H	G	I	E	D	A	K
I	O	G	B	C	L	A	J	D	N	M	H	E	K	F
B	L	I	F	A	J	D	K	O	M	C	N	G	E	H
J	M	L	K	F	C	H	A	N	I	B	G	O	D	E

Hedayat [3] has shown that this latin square can be embedded in a set of three mutually orthogonal latin squares of order 15, thus disproving MacNeish's conjecture [4] for this order. The other two latin squares are exhibited below:

D	I	E	A	O	K	C	H	M	G	B	L	N	J	F
L	H	N	G	F	E	B	M	O	J	K	A	C	D	I
O	N	J	I	K	G	L	C	B	H	A	F	D	M	E
N	A	F	B	D	O	H	G	J	K	M	C	I	E	L
J	F	K	L	M	B	N	I	H	A	C	D	E	G	O
C	G	I	K	L	F	M	J	D	E	O	B	H	N	A
E	L	O	C	J	M	I	A	G	B	H	N	F	K	D
H	K	A	O	C	L	J	E	N	F	D	M	G	I	B
F	C	D	H	E	J	A	B	K	M	G	I	L	O	N
B	J	H	E	A	I	O	D	L	N	F	K	M	C	G
M	E	G	F	B	A	D	N	C	I	L	O	J	H	K
A	M	C	J	I	N	K	O	F	D	E	G	B	L	H
K	D	M	N	G	H	F	L	A	C	I	E	O	B	J
G	O	B	D	H	C	E	F	I	L	N	J	K	A	M
I	B	L	M	N	D	G	K	E	O	J	H	A	F	C

I	C	D	H	F	N	K	A	J	L	O	G	M	B	E
A	O	G	J	C	K	D	L	I	H	E	B	F	N	M
B	J	E	D	H	C	O	N	G	K	I	M	L	F	A
O	N	H	F	B	A	L	E	C	M	K	J	G	I	D
J	F	K	L	M	B	N	I	H	A	C	D	E	G	O
E	K	J	C	N	L	H	M	O	F	D	I	B	A	G
C	B	L	M	G	I	A	O	F	E	N	H	D	J	K
D	L	C	N	A	M	F	G	K	B	H	O	I	E	J
N	I	M	G	J	O	E	K	B	D	A	F	H	C	L
M	D	I	O	K	H	J	F	E	C	G	A	N	L	B
G	H	O	K	L	F	C	B	D	N	J	E	A	M	I
L	E	B	I	O	D	G	H	A	J	M	N	C	K	F
F	G	A	E	D	J	I	C	M	O	B	L	K	H	N
H	M	N	A	E	G	B	J	L	I	F	K	O	D	C
K	A	F	B	I	E	M	D	N	G	L	C	J	O	H

It is easy to verify that the preceding three latin squares forms a set of three mutually orthogonal latin squares of order 15.

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