ANALYSES FOR A 5x5 LATTICE SQUARE IN 2 REPLICATES AND IN 4 REPLICATES

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A time and motion study on the milking of cows was conducted by the Department of Agricultural Engineering. Enough experimental material was available for $k^2 = 25$ treatments with four replicates on each treatment. It was suspected that the error mean square for the first two replicates might be larger than for the last two replicates. The first two replicates might need to be analyzed separately from the last two replicates. Also, since only a portion of the treatments could be carried out each day and since the workers tired as the day progressed a lattice square design was selected; the day to day differences were confounded with row effects and the period to period differences within days were confounded with column effects. The following scheme of confounding of pseudo-effects was used in the 4 replicates:

	Replicate			
Effect confounded with	I	II	III	IV
Rows (days)	A	AB	AB^3	B
Columns (periods)	B	AB^2	AB^4	A

A separate analysis may be performed on the first 2 replicates and on the last 2 replicates. Also, an analysis of variance may be performed on the 4 replicates combined.

The breakdown of the total degrees of freedom in the analysis of variance for replicates I and II, using the totals given in table 1, is given in table 2.

The total $(A)_{.0}$ is obtained from 10 individual yields; thus $(A)_{.0} = X_{100} + X_{200} + X_{101} + X_{201} + X_{102} + X_{202} + X_{103} + X_{203} + X_{104} + X_{204}$. The other

$(A)_{.u}$ are obtained similarly. The differences $(A)_{.u} - 2(A)_{1u} = (A)_{2u} - (A)_{1u}$ represent the effect of the uth level of the pseudo-effect A in the replicate in which it is unconfounded (versus the same effect in the replicate in which it is unconfounded with row differences. The other differences are explainable in a similar manner. The sums of squares of these differences within replicates comprise the sums of squares for rows (eliminating treatments) and for columns (eliminating treatments).

The intra-row and -column sum of squares may be obtained by subtraction or by direct computations. It is made up of the interaction of levels of AB^3 and replicates and the interaction of levels of AB^4 effects and replicates. This sum of squares is associated with 4+4=8 degrees of freedom.

The weights of w, w_r, and w_c are estimated from the observed mean squares (table 2) as follows:

$$w = 1/E_e,$$

$$w_r = 1/(2E_r - E_e),$$

and

$$w_c = 1/(2E_c - E_e).$$

The average standard error of a difference between 2 adjusted means is equal to

$$\sqrt{\frac{2}{6}\left\{\frac{2}{w_r + w} + \frac{2}{w_c + w} + \frac{2}{2w}\right\}} = \sqrt{\frac{2}{3}\left\{\frac{1}{w_r + w} + \frac{1}{w_c + w} + \frac{1}{2w}\right\}}.$$

The adjustments for variety means are obtained by adding 4 adjustments (2 for rows and 2 for columns) to the unadjusted variety mean. The row adjustments are obtained by multiplying the differences $(A)_{.u} - 2(A)_{1u}$ and $(AB)_{.u} - 2(AB)_{2u}$ by the factor $\frac{w - w_r}{2k(w + w_r)}$. The column adjustments are obtained similarly in that the differences $(B)_{.u} - 2(B)_{1u}$ and $(AB^2)_{.u} - 2(AB^2)_{2u}$ are multiplied by $\frac{w - w_c}{2k(w + w_c)}$. To illustrate, the adjusted mean for treatment 01 is equal to the unadjusted mean $+ \frac{(w - w_r)}{2(k)(w + w_r)} \left\{ (A)_{.0} - 2(A)_{10} + (AB)_{.1} - 2(AB)_{21} \right\} + \frac{w - w_c}{2(k)(w + w_c)} \left\{ (B)_{.1} - 2(B)_{11} + (AB^2)_{.2} - 2(AB^2)_{22} \right\}$.

The average effective error variance is equal to the square of the average standard error of mean difference in this particular case. The estimated error mean square for a randomized complete block, E_e', divided by the average effective error variance is a measure of the efficiency of the design; thus, efficiency $= E_e' (100) \Big/ \frac{2}{3} \left\{ \frac{1}{w_r + w} + \frac{1}{w_c + w} + \frac{1}{2w} \right\}$.

The analysis for replicates III and IV follows that outlined above for replicates I and II.

The analysis of variance, standard errors, adjustments for treatment means, etc. for the 4 replicates combined presents somewhat greater difficulties. If the notation of table 1 is extended to include all 4 replicates the appropriate sums of squares for the analysis of variance table are presented in table 3 for 25 treatments. As a check the sum of the sums of squares for row (eliminating treatment and ignoring column effects) and column (eliminating both treatment and row effects) should be equal to the sum of the sums of squares for row (eliminating both treatment and column effects) and column (eliminating treatment and ignoring row effects).

The intra-row and -column error sum of squares may be obtained by subtraction, or directly as the sum of the following sums of squares:

Interaction	of	levels	of	A	with replicates	II	and	III	with	4df
"	"	"	"	B	" "	II	"	III	"	4df
"	"	"	"	AB	" "	I,III	"	IV	"	8df
"	"	"	"	AB^2	" "	I,III	"	IV	"	8df
"	"	"	"	AB^3	" "	I,II	"	IV	"	8df
"	"	"	"	AB^4	" "	I,II	"	IV	"	8df
										40df

The weights are estimated from the mean squares in table 3; thus,

$$w = 1/E'_e,$$

$$w_r = \frac{17}{24E_r - 7E_e},$$

and

$$w_c = \frac{17}{24E_c - 7E_e}.$$

The average standard error of a difference between 2 adjusted means equals

$$\sqrt{\frac{2}{3}\left\{\frac{1}{w_r + w_c + 2w} + \frac{1}{w_r + 3w} + \frac{1}{w_c + 3w}\right\}}.$$

To obtain adjustments for treatment means multiply the differences $(A)_{.u} - 4(A)_{1u}$ and $(B)_{.u} - 4(B)_{4u}$ by the factor $\frac{w - w_r}{4(5)(w_r + w_c + 2w)}$; multiply the differences $(A)_{.u} - 4(A)_{4u}$ and $(B)_{.u} - 4(B)_{1u}$ by the factor $\frac{w - w_c}{4(5)(w_r + w_c + 2w)}$; multiply the differences $(AB)_{.u} - 4(AB)_{2u}$ and $(AB^3)_{.u} - 4(AB^3)_{3u}$ by the factor $\frac{w - w_r}{4(5)(w_r + 3w)}$; and multiply the differences $(AB^2)_{.u} - 4(AB^2)_{2u}$ and $(AB^4)_{.u} - 4(AB^4)_{3u}$ by the factor $\frac{w - w_c}{4(5)(w_c + 3w)}$. The sum of the corresponding 4 row and 4 column adjustments for each treatment is added to the unadjusted treatment mean to obtain the adjusted treatment mean. As in the previous case with 2 replicates the sum of the adjusted treatment means should equal the sum of the unadjusted treatment means. This requires that the sum total of the adjustments must equal zero.

The efficiency of this lattice square relative to a randomized complete block design is

$$E_e^1 \Big/ \frac{4}{3}\left\{\frac{1}{w_r + w_c + 2w} + \frac{1}{w_r + 3w} + \frac{1}{w_c + 3w}\right\}$$

per cent.

Table 1. Symbolic representation of yield and totals for a 5 x 5 lattice square

Replicate I (A and B confounded)

						Total
	X_{100}	X_{101}	X_{102}	X_{103}	X_{104}	$X_{10\cdot} = (A)_{10}$
	X_{110}	X_{111}	X_{112}	X_{113}	X_{114}	$X_{11\cdot} = (A)_{11}$
	X_{120}	X_{121}	X_{122}	X_{123}	X_{124}	$X_{12\cdot} = (A)_{12}$
	X_{130}	X_{131}	X_{132}	X_{133}	X_{134}	$X_{13\cdot} = (A)_{13}$
	X_{140}	X_{141}	X_{142}	X_{143}	X_{144}	$X_{14\cdot} = (A)_{14}$
Total	$(B)_{10}$	$(B)_{11}$	$(B)_{12}$	$(B)_{13}$	$(B)_{14}$	$X_{1\cdot\cdot}$

Replicate II (AB and AB^2 confounded)

						Total
	X_{200}	X_{241}	X_{232}	X_{223}	X_{214}	$(AB)_{20}$
	X_{224}	X_{210}	X_{201}	X_{242}	X_{233}	$(AB)_{21}$
	X_{243}	X_{234}	X_{220}	X_{211}	X_{202}	$(AB)_{22}$
	X_{212}	X_{203}	X_{244}	X_{230}	X_{221}	$(AB)_{23}$
	X_{231}	X_{222}	X_{213}	X_{204}	X_{240}	$(AB)_{24}$
Total	$(AB^2)_{20}$	$(AB^2)_{21}$	$(AB^2)_{22}$	$(AB^2)_{23}$	$(AB^2)_{24}$	$X_{2\cdot\cdot}$

Replicate III (AB^3 and AB^4 confounded)

						Total
	X_{300}	X_{321}	X_{342}	X_{313}	X_{334}	$(AB^3)_{30}$
	X_{344}	X_{310}	X_{331}	X_{302}	X_{323}	$(AB^3)_{31}$
	X_{333}	X_{304}	X_{320}	X_{341}	X_{312}	$(AB^3)_{32}$
	X_{322}	X_{343}	X_{314}	X_{330}	X_{301}	$(AB^3)_{33}$
	X_{311}	X_{332}	X_{303}	X_{324}	X_{340}	$(AB^3)_{34}$
Total	$(AB^4)_{30}$	$(AB^4)_{31}$	$(AB^4)_{32}$	$(AB^4)_{33}$	$(AB^4)_{34}$	$X_{3\cdot\cdot}$

Replicate IV (B and A confounded)

						Total
	X_{400}	X_{410}	X_{420}	X_{430}	X_{440}	$(B)_{40} = X_{40\cdot}$
	X_{401}	X_{411}	X_{421}	X_{431}	X_{441}	$(B)_{41} = X_{41\cdot}$
	X_{402}	X_{412}	X_{422}	X_{432}	X_{442}	$(B)_{42} = X_{42\cdot}$
	X_{403}	X_{413}	X_{423}	X_{433}	X_{443}	$(B)_{43} = X_{43\cdot}$
	X_{404}	X_{414}	X_{424}	X_{434}	X_{444}	$(B)_{44} = X_{44\cdot}$
Total	$(A)_{40}$	$(A)_{41}$	$(A)_{42}$	$(A)_{43}$	$(A)_{44}$	$X_{4\cdot\cdot}$

Table 2 Analysis of variance for a kxk lattice square in 2 replicates.

Source of variation	df	Sum of squares	Mean square Observed	Mean square Expected value
Replicate	1	$\frac{X_{1..}^2+X_{2..}^2}{k^2} - \frac{X_{...}^2}{2k^2}$	-	-
Treatment (ignoring row and column effects)	k^2-1	$\sum\sum \frac{X_{.ij}^2}{2} - \frac{X_{...}^2}{2k^2}$	-	-
Remainder = estimate of rand. complete bl. error	k^2-1	By subtraction	E_c'	-
Row (eliminating treatment)	$2(k-1)$	$\sum_{u=0}^{k-1}\left\{\frac{[(A)_{.u}-2(A)_{1u}]^2+[(AB)_{.u}-2(AB)_{2u}]^2}{2k}\right\} - \frac{2(X_{1..}-X_{2..})^2}{2k^2}$	E_r	$\sigma_\varepsilon^2+k\sigma_\rho^2/2$
Column (eliminating treatment)	$2(k-1)$	$\sum_{u=0}^{k-1}\left\{\frac{[(B)_{.u}-2(B)_{1u}]^2+[(AB^2)_{.u}-2(AB^2)_{2u}]^2}{2k}\right\} - \frac{2(X_{1..}-X_{2..})^2}{2k^2}$	E_c	$\sigma_\varepsilon^2+k\sigma_\gamma^2/2$
Intrablock error	$(k-1)(k-3)$	By subtraction or by direct computation as the interaction of levels of effects $AB^3,\ldots,AB^{k-1}$ with replicates	E_e	σ_ε^2
Total	$2k^2-1$	$\sum\sum\sum X_{gij}^2 - X_{...}^2/2k^2$		

Table 3 Analysis of variance for 4 replicates of a 5x5 lattice square.

Source of variation	df	Sum of squares	Mean square: Observed	Mean square: Expected value
Replicate	3	$\Sigma X^2_{g..}/25 - X^2_{...}/100$	-	-
Treatment (ign.row and col.)	24	$\Sigma\Sigma X^2_{.ij}/4 - X^2_{...}/100$	-	-
Remainder = est.of randomized bl.error	72	By subtraction	E'_e	-
Row (elim.tr.; ign. col.)	16	$\frac{1}{60}\Sigma\{[(A)_{.u}-4(A)_{1u}]^2+[(AB)_{.u}-4(AB)_{2u}]^2+[(AB^3)_{.u}-4(AB^3)_{3u}]^2+[(B)_{.u}-4(B)_{4u}]^2\}$ $-[(X_{...}-4X_{1..})^2+(X_{...}-4X_{2..})^2+(X_{...}-4X_{3..})^2+(X_{...}-4X_{4..})^2]/300$	-	-
Column (elim.tr. and row)	16	$\frac{1}{30}\Sigma\{[(A)_{.u}-3(A)_{4u}-(A)_{1u}]^2+[(B)_{.u}-3(B)_{1u}-(B)_{4u}]^2+\frac{1}{60}\Sigma\{[(AB^2)_{.u}-4(AB^2)_{2u}]^2+[(AB^4)_{.u}-4(AB^4)_{3u}]^2\}$ $-[(X_{2..}+X_{3..}-2X_{1..})^2+(X_{2..}+X_{3..}-2X_{4..})^2]/150-[(X_{...}-4X_{2..})^2+(X_{...}-4X_{3..})^2]/300$	E_c	$\sigma^2_\varepsilon+85\sigma^2_\gamma/24$
Intrablock	40	By subtraction or by direct computation of levels of effects with replicates in which the effects are unconfounded	E_e	σ^2_ε
Total	99	$\Sigma\Sigma\Sigma X^2_{gij} - X^2_{...}/100$	-	-
Row (elim. tr. and col.)	16	$\frac{1}{30}\Sigma\{[(A)_{.u}-3(A)_{1u}-(A)_{4u}]^2+[(B)_{.u}-3(B)_{4u}-(B)_{1u}]^2+\frac{1}{60}\Sigma\{[(AB)_{.u}-4(AB)_{2u}]^2+[(AB^3)_{.u}-4(AB^3)_{3u}]^2\}$ $-\frac{1}{150}\{(X_{2..}+X_{3..}-2X_{1..})^2+(X_{2..}+X_{3..}-2X_{4..})^2\}-\frac{1}{300}\{(X_{...}-4X_{2..})^2+(X_{...}-4X_{3..})^2\}$	E_r	$\sigma^2_\varepsilon+85\sigma^2_\rho/24$
Column (elim.tr.; ignoring row)	16	$\Sigma\{[(B)_{.u}-4(B)_{1u}]^2+[(AB^2)_{.u}-4(AB^2)_{2u}]^2+[(AB^4)_{.u}-4(AB^4)_{3u}]^2+[(A)_{.u}-4(A)_{4u}]^2\}/60$ $-\{(X_{...}-4X_{1..})^2+(X_{...}-4X_{2..})^2+(X_{...}-4X_{3..})^2+(X_{...}-4X_{4..})^2\}/300$	-	-

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