VARIANCE COMPONENT ESTIMATION: PROOF OF EQUIVALENCE
OF ALTERNATIVE METHOD 4 TO HENDERSON'S METHOD 3

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Abstract

An alternative way of carrying out the generalized form of
Henderson's Method 2 for estimating variance components is shown
equivalent to Henderson's Method 3.

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Zelen, in the discussion of Searle (1968) states that alternative
Method 4 given there is equivalent to Method 3. Proof of this is now
given.

The model being considered is

\[ y = \mu + X_1 \beta_1 + X_2 \beta_2 + X_3 \beta_3 + X_4 \beta_4 + e \]

We assume that \( \beta_f = \beta_1 \) and \( \beta_r = (\beta_2 \ \beta_3 \ \beta_4) \) are the fixed and random
effects respectively. Now in Method 3, the sum of squares for fitting
\( \beta_3 \) alone would be

\[ R(\beta_3) = y'X_3(X_3'X_3)^{-1}X_3'y. \] (1)

The alternative Method 4 is to apply Method 3 directly to \( z \), where

\[ z = Wy \] (2)

\[ W = I - X_f(X_f'X_f)^{-1}X_f' \] (3)

\[ W = W' = W^2. \] (4)

The model for \( z \) is

\[ z = WX_1 \beta_f + We \]

\[ = WX_2 \beta_2 + WX_3 \beta_3 + WX_4 \beta_4 + We. \] (5)
Therefore, applying (1) to this, we get, for fitting just $\beta_3$ alone (i.e. reduced model $z = WX_3^2 + W_3$)

$$R(\beta_3) = z'(WX_3)(WX_3)'z$$

$$= z'WX_3X_3'WX_3 - X_3'W_3z$$

$$= y'W^2X_3X_3'X_3'W^2y$$

$$= y'WX_3X_3'X_3'Wy.$$

To show that this is identical to applying generalized least squares to the model $z = WX_3^2 + W_3$, we first recall generalized least squares for the general model

$$y = X\beta + e, \text{ with } \text{var}(e) = V.$$  

The sum of squares is

$$R(\beta) = y'V^{-1}X(X'V^{-1}X)'V^{-1}y.$$  

The analogy with $z = WX_3^2 + W_3$, where $\text{var}(W_3) = W$ is

$$R(\beta_3)_{\text{GLS}} = z'W^{-1}(WX_3)(WX_3)'W^{-1}z$$

$$= y'WW^{-1}WX_3X_3'WW^{-1}Wy$$

$$= y'WX_3X_3'X_3'Wy$$

$$= R(\beta_3).$$

Therefore the alternative Method 4 (Method 5) is equivalent to generalized least squares.
We now show that this Method is also equivalent to Method 3 applied to \( y \). To do this we utilize, from (1)

\[
R(\beta_3|\beta_f) = y'X_f(X_f'X_f)^{-1}X_f'y
\]

and so

\[
R(\beta_3|\beta_f) = R(\beta_f|\beta_3) - R(\beta_f)_3
\]

Write

\[
Q = X_3'X_3 - X_3'X_f(x_f'x_f)^{-1}x_f'X_3 = X_3'X_3
\]

Then from the generalized inverse of a partitioned matrix

\[
R(\beta_3|\beta_f) = y'(X_f'X_3)
\]

\[
\begin{bmatrix}
(x_f'x_f)^{-1} - Q^{-1}x_f'x_f(x_f'x_f)^{-1} - Q^{-1}x_f'X_3
- Q^{-1}X_f(x_f'x_f)^{-1}
- Q^{-1}X_f
0
0
\end{bmatrix} \begin{bmatrix}
X_f'
y
\end{bmatrix}
\]

\[
= y'(X_f'X_3)
\]

\[
\begin{bmatrix}
(x_f'x_f)^{-1} - I
-I
\end{bmatrix} Q^{-1}[x_f'x_f(x_f'x_f)^{-1} - x_f'x_3] y
\]

\[
= y'[x_f(x_f'x_f)^{-1}x_f'x_3 - x_f] Q^{-1}[x_f'x_f(x_f'x_f)^{-1} - x_f'] y
\]
Hence Method 3, on \( z \), gives \( R(s_3) \) which is identical to Method 3 on \( y \) giving \( R(\beta_3|\beta_f) \). This seems to suggest some kind of optimality for Method 3.

Reference