EXPECTATION OF MEAN SQUARES FROM AN EXPERIMENT FOR PERENNIAL CROPS

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Given the following lay-out and yields X_{ghij}

	Hay Cut			Aftermath			
	Replicates						
Year 1 Varieties	1	2 ... r		1	2 ... r		
1	X_{1111}	$X_{1121} \cdots X_{11r1}$	$X_{11\cdot 1}$	X_{1211}	$X_{1221} \cdots X_{12r1}$	$X_{12\cdot 1}$	$X_{1\cdot\cdot 1}$
2	X_{1112}	$X_{1122} \cdots X_{11r2}$	$X_{11\cdot 2}$	X_{1212}	$X_{1222} \cdots X_{12r2}$	$X_{12\cdot 2}$	$X_{1\cdot\cdot 2}$
⋮							
v	X_{111v}	$X_{112v} \cdots X_{11rv}$	$X_{11\cdot v}$	X_{121v}	$X_{122v} \cdots X_{12rv}$	$X_{12\cdot v}$	$X_{1\cdot\cdot v}$
	$X_{111\cdot}$	$X_{112\cdot} \cdots X_{11r\cdot}$	$X_{11\cdot\cdot}$	$X_{121\cdot}$	$X_{122\cdot} \cdots X_{12r\cdot}$	$X_{12\cdot\cdot}$	$X_{1\cdot\cdot\cdot}$

Year 2

⋮

Year y

and the linear model

$$X_{ghij} = \mu + \alpha_g + \gamma_h + (\alpha\gamma)_{gh} + \rho_i + (\alpha\rho)_{gi} + (\gamma\rho)_{hi} + (\alpha\gamma\rho)_{ghi} + \tau_j + (\rho\tau)_{ij} + (\alpha\tau)_{gj} + (\gamma\tau)_{hj} + (\alpha\gamma\tau)_{ghj} + (\gamma\rho\tau)_{hij} + (\alpha\rho\tau)_{gij} + \varepsilon_{ghij}$$

where μ = mean effect,

α_g = effect due to gth year,

γ_h = effect of hth cut,

$(\alpha\gamma)_{gh}$ = effect peculiar to hth cut in gth year,

ρ_i = effect due to ith replicate,

$(\alpha\rho)_{gi}$ = effect due to ith replicate in gth year,

$(\gamma\rho)_{hi}$ = effect due to hth cut in ith replicate,

$(\alpha\gamma\rho)_{ghi}$ = effect due to hth cut in ith replicate in the gth year,

τ_j = effect due to jth variety,

$(\rho\tau)_{ij}$ = effect due to jth variety in ith replicate,

$(\alpha\tau)_{gj}$ = effect due to jth variety in gth year,

$(\gamma\tau)_{hj}$ = effect due to jth variety in hth cut,

$(\alpha\gamma\tau)_{ghj}$ = effect due to jth variety in hth cut in the gth year,

$(\gamma\rho\tau)_{hij}$ = effect of hth cut on jth variety in the ith replicate,

$(\alpha\rho\tau)_{gij}$ = effect of jth variety in the ith replicate in the gth year,

ε_{ghij} = effect common to jth variety in hth cut in the ith replicate in the gth year.

The analysis of variance table for the above is:

Source of variation	d.f.	Sum of squares
Years	y-1	$\sum_g X^2_{g\cdots}/vcr - X^2_{\cdots\cdot}/rvcy = Y$
Cuts	c-1	$\sum_h X^2_{\cdot h\cdot\cdot}/rvy - X^2_{\cdots\cdot}/crvy = C$
Years x Cuts	(y-1)(c-1)	$\sum\sum_{gh} X^2_{gh\cdot\cdot}/rv - Y - C - X^2_{\cdots\cdot}/crvy = YC$
Replicates	r-1	$\sum_i X^2_{\cdot\cdot i\cdot}/cvy - X^2_{\cdots\cdot}/crvy = R$
Replicate x Year	(r-1)(y-1)	$\sum\sum_{gi} X^2_{g\cdot i\cdot}/cv - X^2_{\cdots\cdot}/crvy - R - Y = RY$
Replicate x Cut	(r-1)(c-1)	$\sum\sum_{hi} X^2_{\cdot hi\cdot}/vy - X^2_{\cdots\cdot}/crvy - R - C = CY$
Rep x Cut x Yr	(r-1)(y-1)(c-1)	$\sum\sum\sum_{ghi} X^2_{ghi\cdot}/v - X^2_{\cdots\cdot}/crvy - R - C - Y - RC - RY - CY = RCY$
Varieties	v-1	$\sum_j X^2_{\cdots j}/cry - X^2_{\cdots\cdot}/crvy = V$
Var x Rep	(v-1)(r-1)	$\sum\sum_{ij} X^2_{\cdot\cdot ij} - X^2_{\cdots\cdot}/crvy - V - R = VR$
Var x Year	(v-1)(y-1)	$\sum\sum_{gj} X^2_{g\cdot\cdot j}/rc - X^2_{\cdots\cdot}/crvy - V - Y = VY$
Var x Cut	(v-1)(c-1)	$\sum\sum_{hj} X^2_{\cdot h\cdot j}/ry - X^2_{\cdots\cdot}/crvy - V - C = VC$

Analysis of variance table (continued)

Var x Cut x Year	(v-1)(c-1)(y-1)	$\sum\sum\sum_{ghj} X^2_{gh\bullet j}/r - X^2_{\bullet\bullet\bullet\bullet}/crvy - V - C - Y - VC - VY - CY = VCY$
Var x Cut x Rep	(v-1)(c-1)(r-1)	$\sum\sum\sum_{hij} X^2_{\bullet hij}/y - X^2_{\bullet\bullet\bullet\bullet}/crvy - V - C - R - VC - VR - CR = VCR$
Var x Year x Rep	(v-1)(y-1)(r-1)	$\sum\sum\sum_{gij} X^2_{g\bullet ij}/c - X^2_{\bullet\bullet\bullet\bullet}/crvy - V - Y - R - VR - VY - RY = VYR$
Residual	(v-1)(y-1)(c-1)(r-1)	$\sum\sum\sum\sum X^2_{ghij} - X^2_{\bullet\bullet\bullet\bullet}/crvy$ - all sums of squares above

The above sums of squares are those for a randomized complete blocks experiment on a perennial crop conducted over y years with c cuttings per year. In most instances, the number of cuttings obtained from a forage crop species in a given area is determined by the growth behaviour of the species in that locality. Alfalfa, for example, may be managed on a one, two, three or four cut system depending on the variety used and where it is grown. In exceptional years, one cut more or less than the standard management practice for an area might be obtained due to favorable or adverse weather conditions, although generally, the experimenter must follow a cutting system predetermined by climatic conditions in his area and is not at liberty to alter efficiency by increasing or decreasing number of cuttings.

In most cutting systems in the North East where forage crops are being used for hay, the first cutting will produce more than one half of the total yield. Subsequent production after the first cut is removed usually does not equal the first cut yield regardless of whether one or more cuttings are made. This same trend is followed in silage and pasture management -- the majority of the yield being produced in the early summer. For practical purposes only two crops are removed each year, the heavy early spring growth and the subsequent recovery growth designated as aftermath production.

Aftermath production is harvested in one to several cuts dependent upon the species, the season, and the way it is to be utilized. Since seasonal variation is of considerable magnitude, the combined aftermath yield, if more than one cut is made, is a practical and convenient way of expressing the yield obtained after the first cut is removed. Actually, then, there are only 2 cuttings per year, first cut on hay cutting and the aftermath cuttings, for the experiments under consideration, but the formulas are developed for c cuttings per year for more general application.

The mean effect μ is considered to be a constant. The number of cuts in these experiments constitute the whole of the population, i.e., there are only c types of individuals in the population of cuttings. Therefore, $E\mu = \mu$, $\sum_h \gamma_h = 0 = \sum_h (\alpha\gamma)_{gh} = \sum_h (\gamma\rho)_{hi} = \sum_h (\gamma\tau)_{gj}$, $E\gamma_h^2 = \gamma_h^2$, $E(\alpha\gamma)_{gh}^2 = \sigma_{\alpha\gamma}^2$, $E(\gamma\rho)_{hi}^2 = \sigma_{\gamma\rho}^2$, and $E(\gamma\tau)_{gj}^2 = \sigma_{\gamma\tau}^2$. In order to determine which effects sum to zero when summed over cuts, Table 1 was prepared. A complete enumeration of the effects in the population is indicated for the three two-factor interaction effects and for one, year x variety x cut, of the three three-factor interaction effects involving cuts; the remaining two three-factor interaction effects, year x cut x replicate and cut x replicate x variety, may be enumerated in a similar manner. From Table 1, then, it will be observed that the following summations equal zero:

$$\sum_h (\alpha\gamma\rho)_{ghj} = \sum_h (\alpha\gamma\tau)_{ghi} = \sum_h (\gamma\rho\tau)_{hij} = 0$$

The following expectations are assumed:

$$E(\alpha\gamma\rho)_{ghj}^2 = \sigma_{\alpha\gamma\rho}^2, \quad E(\alpha\gamma\tau)_{ghi}^2 = \sigma_{\alpha\gamma\tau}^2, \quad \text{and} \quad E(\gamma\rho\tau)_{hij}^2 = \sigma_{\gamma\rho\tau}^2 .$$

The remaining effects are considered to be random independent variables from an infinitely large population with zero means and variances peculiar to each effect.

The average value of the expected values or summed values of all cross products equals zero.

Since the sample of years represents consecutive years, it might be argued that they do not represent independent variates. Since it is doubtful if the dependence is of any appreciable size and since an experiment on perennial forage crops will never be conducted any other way, the sample of years is regarded as a random independent sample of years.

Cuts, on the other hand, were not regarded as independent. Regardless of the desires of the experimenter, there will only be hay yields and aftermath yields on some types of forage crops experiments. Therefore it would be idle to speculate about the relative efficiency of increasing the number of cuttings per year. Two cuts will be available each year. It is possible on perennials but not on biennials to increase the number of years on which yields are taken. However it is doubtful if yields will be obtained for more than 4 to 6 years at most.

Based on the above considerations, the expectation of the total sum of squares with rcvy degress of freedom is obtained as follows:

$$E\left[\sum_g \sum_h \sum_i \sum_j X^2_{ghij}\right] = \sum_g \sum_h \sum_i \sum_j E\left[X^2_{ghij}\right] = \sum\sum\sum\sum E\left[\mu + \alpha_g + \gamma_h + (\alpha\gamma)_{gh} + \rho_i + (\alpha\rho)_{gi} + (\gamma\rho)_{hi} + (\alpha\gamma\rho)_{ghi} + \tau_j + (\alpha\tau)_{gj} + (\gamma\tau)_{hj} + (\rho\tau)_{ij} + (\alpha\gamma\tau)_{ghj} + (\alpha\rho\tau)_{gij} + (\gamma\rho\tau)_{hij} + \varepsilon_{ghij}\right]^2 = rvy\sum\gamma^2_h + rcvy\left[\mu^2 + \sigma^2_\alpha + \sigma^2_{\alpha\gamma} + \sigma^2_\rho + \sigma^2_{\alpha\rho} + \sigma^2_{\gamma\rho} + \sigma^2_{\alpha\gamma\rho} + \sigma^2_\tau + \sigma^2_{\alpha\tau} + \sigma^2_{\gamma\tau} + \sigma^2_{\rho\tau} + \sigma^2_{\alpha\gamma\tau} + \sigma^2_{\alpha\rho\tau} + \sigma^2_{\gamma\rho\tau} + \sigma^2_\varepsilon\right].$$

Table 1. Enumeration of interaction effects in the population

	Year x cut effects						
	g =						
h=	1	2	...	y	...	∞	Sum
1	$(\alpha\gamma)_{11}$	$(\alpha\gamma)_{21}$		$(\alpha\gamma)_{y1}$		$(\alpha\gamma)_{\infty 1}$	0
2	$(\alpha\gamma)_{12}$	$(\alpha\gamma)_{22}$		$(\alpha\gamma)_{y2}$		$(\alpha\gamma)_{\infty 2}$	0
⋮	⋮	⋮		⋮		⋮	⋮
c	$(\alpha\gamma)_{1c}$	$(\alpha\gamma)_{2c}$		$(\alpha\gamma)_{yc}$		$(\alpha\gamma)_{\infty c}$	0
Sum	0	0	...	0	...	0	0

	Replicate x cut effects						
	j =						
h=	1	2	...	r	...	∞	Sum
1	$(\gamma\rho)_{11}$	$(\gamma\rho)_{21}$		$(\gamma\rho)_{r1}$		$(\gamma\rho)_{\infty 1}$	0
2	$(\gamma\rho)_{12}$	$(\gamma\rho)_{22}$		$(\gamma\rho)_{r2}$		$(\gamma\rho)_{\infty 2}$	0
⋮	⋮	⋮		⋮		⋮	⋮
c	$(\gamma\rho)_{1c}$	$(\gamma\rho)_{2c}$		$(\gamma\rho)_{rc}$		$(\gamma\rho)_{\infty c}$	0
Sum	0	0	...	0	...	0	0

	Variety x cut effects						
	i =						
h=	1	2	...	v	...	∞	Sum
1	$(\gamma\tau)_{11}$	$(\gamma\tau)_{21}$		$(\gamma\tau)_{v1}$		$(\gamma\tau)_{\infty 1}$	0
2	$(\gamma\tau)_{12}$	$(\gamma\tau)_{22}$		$(\gamma\tau)_{v2}$		$(\gamma\tau)_{\infty 2}$	0
⋮	⋮	⋮		⋮		⋮	⋮
c	$(\gamma\tau)_{1c}$	$(\gamma\tau)_{2c}$		$(\gamma\tau)_{vc}$		$(\gamma\tau)_{\infty c}$	0
Sum	0	0	...	0	...	0	0

	Year x cut x variety effects							
	g = 1							...
	j =							
h=	1	...	r	...	∞	Sum		
1	$(\alpha\gamma\tau)_{111}$		$(\alpha\gamma\tau)_{11r}$		$(\alpha\gamma\tau)_{11\infty}$	0		
2	$(\alpha\gamma\tau)_{121}$		$(\alpha\gamma\tau)_{12r}$		$(\alpha\gamma\tau)_{12\infty}$	0		
⋮	⋮		⋮		⋮	⋮		
c	$(\alpha\gamma\tau)_{1c1}$		$(\alpha\gamma\tau)_{1cr}$		$(\alpha\gamma\tau)_{1c\infty}$	0		
Sum	0	...	0	...	0	0		...

	g = y						
	j =						...
h=	1	...	r	...	∞	Sum	
1	$(\alpha\gamma\tau)_{y11}$		$(\alpha\gamma\tau)_{y1r}$		$(\alpha\gamma\tau)_{y1\infty}$	0	
2	$(\alpha\gamma\tau)_{y21}$		$(\alpha\gamma\tau)_{y2r}$		$(\alpha\gamma\tau)_{y2\infty}$	0	
⋮	⋮		⋮		⋮	⋮	
c	$(\alpha\gamma\tau)_{yc1}$		$(\alpha\gamma\tau)_{ycr}$		$(\alpha\gamma\tau)_{yc\infty}$	0	
Sum	0	...	0	...	0	0	...

	g = ∞					
	j =					
h=	1	...	r	...	∞	Sum
1	$(\alpha\gamma\tau)_{\infty 11}$		$(\alpha\gamma\tau)_{\infty 1r}$		$(\alpha\gamma\tau)_{\infty 1\infty}$	0
2	$(\alpha\gamma\tau)_{\infty 21}$		$(\alpha\gamma\tau)_{\infty 2r}$		$(\alpha\gamma\tau)_{\infty 2\infty}$	0
⋮	⋮		⋮		⋮	⋮
c	$(\alpha\gamma\tau)_{\infty c1}$		$(\alpha\gamma\tau)_{\infty cr}$		$(\alpha\gamma\tau)_{\infty c\infty}$	0
Sum	0	...	0	...	0	0

The expectation of the correction term is

$$E[X^2_{\cdots\cdot}/crvy] = \frac{1}{crvy} E[\, crvy\,\mu + crv \sum_g \alpha_g + 0 + 0 + cvy \sum_i \rho_i + cv \sum\sum_{gi} (\alpha\rho)_{gi}$$

$$+ 0 + 0 + cry \sum_j \tau_j + rc \sum\sum_{gj} (\alpha\tau)_{gj} + 0 + cy \sum\sum_{ij} (\rho\tau)_{ij} + 0 + c \sum\sum\sum_{gij} (\alpha\rho\tau)_{gij} + 0$$

$$+ \sum\sum\sum\sum_{ghij} \varepsilon_{ghij}]^2 = crvy\,\mu^2 + crv\,\sigma^2_\alpha + cvy\,\sigma^2_\rho + cv\,\sigma^2_{\alpha\rho} + cry\,\sigma^2_\tau$$

$$+ rc\,\sigma^2_{\alpha\tau} + cy\,\sigma^2_{\rho\tau} + c\sigma^2_{\alpha\rho\tau} + \sigma^2_\varepsilon \,.$$

One degree of freedom is associated with the correction term.

The expectation of the sum of squares of yearly totals is equal to

$$E[\sum_g X^2_{g\cdots}/crv] = E\sum_g \frac{1}{crv} [\, crv\,\mu + crv\,\alpha_g + 0 + 0 + cv \sum_i \rho_i + cv \sum_i (\alpha\rho)_{gi} + 0 + 0$$

$$+ cr \sum_j \tau_j + cr \sum_j (\alpha\tau)_{gj} + 0 + c \sum\sum_{ij} (\rho\tau)_{ij} + 0 + 0 +$$

$$+ c \sum\sum_{ij} (\alpha\rho\tau)_{gij} + \sum\sum\sum_{hij} \varepsilon_{ghij}]^2 = crvy\,(\mu^2 + \sigma^2_\alpha) + cvy\,\sigma^2_\rho$$

$$+ cvy\,\sigma^2_{\alpha\rho} + cry\,\sigma^2_\tau + cry\,\sigma^2_{\alpha\tau} + cy\,\sigma^2_{\rho\tau} + cy\,\sigma^2_{\alpha\rho\tau} + y\,\sigma^2_\varepsilon \,.$$

The expected value of the sum of squares of the cut totals is equal to

$$E[\sum_h X^2_{\cdot h\cdot\cdot}/rvy] = \sum_h \frac{1}{rvy} E[\, rvy\,\mu + rv \sum_g \alpha_g + rvy\,\gamma_h + rv \sum_g (\alpha\gamma)_{gh} + vy \sum_i \rho_i$$

$$+ v \sum\sum_{gi} (\alpha\rho)_{gi} + vy \sum_i (\gamma\rho)_{hi} + v \sum\sum_{gi} (\alpha\gamma\rho)_{ghi} + ry \sum_j \tau_j$$

$$+ y \sum\sum_{ij} (\rho\tau)_{ij} + r \sum\sum_{gj} (\alpha\tau)_{gj} + ry \sum_j (\gamma\tau)_{hj} + r \sum\sum_{gj} (\alpha\gamma\tau)_{ghj}$$

$$+ y \sum\sum_{ij} (\gamma\rho\tau)_{hij} + \sum\sum\sum_{gij} (\alpha\rho\tau)_{gij} + \sum\sum\sum_{gij} \varepsilon_{ghij}]^2 = crvy\,\mu^2$$

$$+ crv\,\sigma^2_\alpha + rvy\,\Sigma\gamma^2_h + crv\,\sigma^2_{\alpha\gamma} + cvy\,\sigma^2_\rho + cv\,\sigma^2_{\alpha\rho} + cvy\,\sigma^2_{\gamma\rho}$$

$$+ cv\ \sigma^2_{\alpha\gamma\rho} + cry\ \sigma^2_{\tau} + cy\ \sigma^2_{\rho\tau} + cr\ \sigma^2_{\alpha\tau} + cry\ \sigma^2_{\gamma\tau} + cr\ \sigma^2_{\alpha\gamma\tau}$$

$$+ cy\ \sigma^2_{\gamma\rho\tau} + c\ \sigma^2_{\alpha\rho\tau} + c\ \sigma^2_{\varepsilon} .$$

The expected value of the sum of squares of totals of years and cuts is equal to $E[\frac{1}{rv}\sum\sum_{gh} X^2_{gh\cdot\cdot}] = \sum\sum_{gh}\frac{1}{rv} E[rv\ (\mu + \alpha_g + \gamma_h + (\alpha\gamma)_{gh}) + v\sum_i\rho_i + \sum_i(\alpha\rho)_{gi}$

$$+ v\sum_i(\gamma\rho)_{hi} + v\sum_i(\alpha\gamma\rho)_{ghi} + r\sum_j\tau_j + \sum\sum_{ij}(\rho\tau)_{ij} + r\sum_j(\alpha\tau)_{gj} + r\sum_j(\gamma\tau)_{hj}$$

$$+ r\sum_j(\alpha\gamma\tau)_{ghj} + \sum\sum_{ij}(\alpha\rho\tau)_{gij} + \sum\sum_{ij}(\gamma\rho\tau)_{hij} + \sum\sum_{ij}\varepsilon_{ghij}]^2 = crvy\ \mu^2 + crvy\sigma^2_{\alpha}$$

$$+ rvy\ \sum\gamma^2_h + crvy\ \sigma^2_{\alpha\gamma} + cvy\ \sigma^2_{\rho} + cvy\ \sigma^2_{\alpha\rho} + cvy\ \sigma^2_{\gamma\rho} + cvy\ \sigma^2_{\alpha\gamma\rho} + rcy\ \sigma^2_{\tau}$$

$$+ rcy\ \sigma^2_{\alpha\tau} + rcy\ \sigma^2_{\gamma\tau} + rcy\ \sigma^2_{\alpha\gamma\tau} + cy\ \sigma^2_{\rho\tau} + cy\ \sigma^2_{\alpha\rho\tau} + cy\ \sigma^2_{\gamma\rho\tau} + cy\ \sigma^2_{\varepsilon} .$$

The expected value of the sum of squares of the replicate totals is equal to $E[\sum_i X^2_{\cdot\cdot i\cdot}/cvy] = \sum_i \frac{1}{cvy} E[cvy\ \mu + cv\sum_g\alpha_g + 0 + 0 + cvy\ \rho_i + cv\sum_g(\alpha\rho)_{gi} + 0$

$$+ 0 + cy\sum_j\tau_j + cy\sum_j(\rho\tau)_{ij} + c\sum\sum_{gj}(\alpha\tau)_{gj} + 0 + 0 + 0$$

$$+ c\sum\sum_{gj}(\alpha\rho\tau)_{gij} + \sum\sum\sum_{ghj}\varepsilon_{ghij}]^2 = crvy\ \mu^2 + crv\ \sigma^2_{\alpha} + rcvy\ \sigma^2_{\rho}$$

$$+ crv\ \sigma^2_{\alpha\rho} + cry\ \sigma^2_{\tau} + cry\ \sigma^2_{\rho\tau} + rc\ \sigma^2_{\alpha\tau} + cr\ \sigma^2_{\alpha\rho\tau} + r\ \sigma^2_{\varepsilon} .$$

The expected value of the sum of squares for years and replicates is equal to $E[\frac{1}{cv}\sum\sum_{gi} X^2_{g\cdot i\cdot}] = \frac{1}{cv}\sum\sum_{gi} E[cv\ \mu + cv\ \alpha_g + 0 + 0 + cv\ \rho_i + cv\ (\alpha\rho)_{gi} + 0 + 0$

$$+ c\sum_j\tau_j + c\sum_j(\rho\tau)_{ij} + c\sum_j(\alpha\tau)_{gj} + 0 + 0 + 0 + c\sum_j(\alpha\rho\tau)_{gij}$$

$$+ \sum\sum_{hj}\varepsilon_{ghij}]^2 = crvy\ \mu^2 + crvy\ \sigma^2_{\alpha} + crvy\ \sigma^2_{\rho} + crvy\ \sigma^2_{\alpha\rho} + cry\ \sigma^2_{\tau}$$

$$+ cry\ \sigma^2_{\rho\tau} + cry\ \sigma^2_{\alpha\tau} + cry\ \sigma^2_{\alpha\rho\tau} + ry\ \sigma^2_{\varepsilon} .$$

The expected value of the sum of squares for replicates and cuts totals is equal to $E[\frac{1}{vy}\sum_{hi}\sum X^2_{\cdot hi \cdot}] = \frac{1}{vy}\sum_{hi}\sum E[vy\,\mu + v\sum_g \alpha_g + vy\,\gamma_h + v\sum_g(\alpha\gamma)_{gh}$

$+ vy\,\rho_i + v\sum_g(\alpha\rho)_{gi} + vy(\gamma\rho)_{hi} + v\sum_g(\alpha\gamma\rho)_{ghi} + y\sum_j \tau_j + y\sum_j(\rho\tau)_{ij}$

$+ \sum_{gj}\sum(\alpha\tau)_{gj} + y\sum_j(\gamma\tau)_{hj} + \sum_{gj}\sum(\alpha\gamma\tau)_{ghj} + y\sum_j(\gamma\rho\tau)_{hij} + \sum_{gj}\sum(\alpha\rho\tau)_{gij}$

$+ \sum_{gj}\sum \varepsilon_{ghij}]^2 = crvy\,\mu^2 + crv\,\sigma^2_\alpha + rvy\,\Sigma\gamma^2_h + crv\,\sigma^2_{\alpha\gamma} + crvy\,\sigma^2_\rho$

$+ crv\,\sigma^2_{\alpha\rho} + crvy\,\sigma^2_{\gamma\rho} + crv\,\sigma^2_{\alpha\gamma\rho} + cry\,\sigma^2_\tau + cry\,\sigma^2_{\rho\tau} + cr\,\sigma^2_{\alpha\tau} + cry\sigma^2_{\gamma\tau}$

$+ cr\,\sigma^2_{\alpha\gamma\tau} + cry\,\sigma^2_{\gamma\rho\tau} + cr\,\sigma^2_{\alpha\rho\tau} + cr\,\sigma^2_\varepsilon\,.$

The expected value of the sum of squares for years, cuts and replicates is equal to $E[\frac{1}{v}\sum_{ghi}\sum\sum X^2_{ghi\cdot}] = \frac{1}{v}\sum_{ghi}\sum\sum E[v\,(\mu + \alpha_g + \gamma_h + (\alpha\gamma)_{gh} + \rho_i + (\alpha\rho)_{gi}$

$+ (\gamma\rho)_{hi} + (\alpha\gamma\rho)_{ghi}) + \sum_j(\tau_j + (\rho\tau)_{ij} + (\alpha\tau)_{gj} + (\gamma\tau)_{hj} + (\alpha\gamma\tau)_{ghj}$

$+ (\gamma\rho\tau)_{hij} + (\alpha\rho\tau)_{gij} + \varepsilon_{ghij})]^2 = crvy\,[\mu^2 + \sigma^2_\alpha + \sigma^2_{\alpha\gamma} + \sigma^2_\rho$

$+ \sigma^2_{\alpha\rho} + \sigma^2_{\gamma\rho} + \sigma^2_{\alpha\gamma\rho}) + vry\,\Sigma\gamma^2_h + cry\,(\sigma^2_\tau + \sigma^2_{\rho\tau} + \sigma^2_{\alpha\tau} + \sigma^2_{\gamma\tau} + \sigma^2_{\alpha\gamma\tau}$

$+ \sigma^2_{\gamma\rho\tau} + \sigma^2_{\alpha\rho\tau} + \sigma^2_\varepsilon)\,.$

The expected sum of squares of the treatment totals is equal to

$E[\frac{1}{cry}\sum_j X^2_{\cdots j}] = \frac{1}{cry}\sum_j E[cry\,\mu + cr\sum_g \alpha_g + 0 + 0 + cy\sum_i \rho_i + c\sum_{gi}\sum(\alpha\rho)_{gi} + 0$

$+ 0 + cry\,\tau_j + cy\sum_i(\rho\tau)_{ij} + cr\sum_g(\alpha\tau)_{gj} + 0 + 0 + 0$

$+ c\sum_{gi}\sum(\alpha\rho\tau)_{gij} + \sum_{ghi}\sum\sum\varepsilon_{ghij}]^2 = crvy\,\mu^2 + crv\,\sigma^2_\alpha + cvy\,\sigma^2_\rho$

$$+ cv\ \sigma^2_{\alpha\rho} + crvy\ \sigma^2_{\tau} + cvy\ \sigma^2_{\rho\tau} + crv\ \sigma^2_{\alpha\tau} + cv\ \sigma^2_{\alpha\rho\tau} + v\ \sigma^2_{\varepsilon}\ .$$

The expected sum of squares for years and varieties in equal to

$$E\left[\frac{1}{cr}\sum_{gj}\sum X^2_{g\cdot\cdot j}\right] = \frac{1}{cr}\sum_{gj}\sum E\left[cr\ (\mu + \alpha_g) + 0 + 0 + c\sum_i \rho_i + c\sum_i(\alpha\rho)_{gi} + 0 + 0\right.$$

$$+ cr\ (\tau_j + (\alpha\tau)_{gj}) + c\sum_i(\rho\tau)_{ij} + 0 + 0 + 0 + c\sum_i(\alpha\rho\tau)_{gij}$$

$$\left. + \sum_{hi}\sum\varepsilon_{ghij}\right]^2 = crvy\ \mu^2 + crvy\ \sigma^2_{\alpha} + cvy\ \sigma^2_{\rho} + cvy\ \sigma^2_{\alpha\rho} + crvy\ \sigma^2_{\tau}$$

$$+ crvy\ \sigma^2_{\alpha\tau} + cvy\ \sigma^2_{\rho\tau} + cvy\ \sigma^2_{\alpha\rho\tau} + vy\ \sigma^2_{\varepsilon}\ .$$

The expected value of the sum of squares of the cuts and varieties totals is equal to $E\left[\frac{1}{ry}\sum_{hj}\sum X^2_{\cdot h\cdot j}\right] = \frac{1}{ry}\sum_{hj}\sum E\left[ry\ \mu + r\sum_g \alpha_g + ry\ \gamma_h + r\sum_g(\alpha\gamma)_{gh}\right.$

$$+ y\sum_i \rho_i + \sum_{gi}\sum(\alpha\rho)_{gi} + y\sum_i(\gamma\rho)_{hi} + \sum_{gi}\sum(\alpha\gamma\rho)_{ghi} + ry\ \tau_j + y\sum_i(\rho\tau)_{ij}$$

$$+ r\sum_g(\alpha\tau)_{gj} + ry(\gamma\tau)_{hj} + r\sum_g(\alpha\gamma\tau)_{ghj} + y\sum_i(\gamma\rho\tau)_{hij} + \sum_{gi}\sum(\alpha\rho\tau)_{gij}$$

$$\left. + \sum_{gi}\sum\varepsilon_{ghij}\right]^2 = crvy\ \mu^2 + crv\ \sigma^2_{\alpha} + rvy\sum\gamma^2_h + crv\ \sigma^2_{\alpha\gamma} + cvy\ \sigma^2_{\rho}$$

$$+ cv\ \sigma^2_{\alpha\rho} + cvy\ \sigma^2_{\gamma\rho} + cv\ \sigma^2_{\alpha\gamma\rho} + crvy\ \sigma^2_{\tau} + cvy\ \sigma^2_{\rho\tau} + crv\ \sigma^2_{\alpha\tau} + crvy\sigma^2_{\gamma\tau}$$

$$+ crv\ \sigma^2_{\alpha\gamma\tau} + cvy\ \sigma^2_{\gamma\rho\tau} + cv\ \sigma^2_{\alpha\rho\tau} + cv\ \sigma^2_{\varepsilon}\ .$$

The expected value of the sum of squares for varieties and replicates is equal to $E\left[\frac{1}{cy}\sum_{ij}\sum X^2_{\cdot\cdot ij}\right] = \frac{1}{cy}\sum_{ij}\sum E\left[cy\ \mu + c\sum_g \alpha_g + 0 + 0 + cy\ \rho_i + c\sum_g(\alpha\rho)_{gi}\right.$

$$+ 0 + 0 + cy\ \tau_j + cy(\rho\tau)_{ij} + c\sum_g(\alpha\tau)_{gj} + 0 + 0 + 0 + c\sum_g(\alpha\rho\tau)_{gij}$$

$$\left. + \sum_{hj}\sum\varepsilon_{ghij}\right]^2 = crvy\ \mu^2 + crv\ \sigma^2_{\alpha} + crvy\ \sigma^2_{\rho} + crv\ \sigma^2_{\alpha\rho} + crvy\ \sigma^2_{\tau} + crvy\sigma^2_{\rho\tau}$$

$$+ crv\ \sigma^2_{\alpha\tau} + crv\ \sigma^2_{\alpha\rho\tau} + rv\ \sigma^2_{\varepsilon}\ .$$

The expected value of the sum of squares for years, cuts and varieties is equal to $E[\frac{1}{r}\sum\sum\sum_{ghj} X^2_{gh\cdot j}] = \frac{1}{r}\sum\sum\sum_{ghj} E[r(\mu + \alpha_g + \gamma_h + (\alpha\gamma)_{gh} + \tau_j + (\alpha\tau)_{gj}$

$+ (\gamma\tau)_{hj} + (\alpha\gamma\tau)_{ghj} + \sum_i(\rho_i + (\alpha\rho)_{gi} + (\gamma\rho)_{hi} + (\alpha\gamma\rho)_{ghi} + (\rho\tau)_{ij}$

$+ (\gamma\rho\tau)_{hij} + (\alpha\rho\tau)_{gij} + \varepsilon_{ghij})]^2 = crvy(\mu^2 + \sigma^2_\alpha + \sigma^2_{\alpha\gamma} + \sigma^2_\tau + \sigma^2_{\alpha\tau}$

$+ \sigma^2_{\gamma\tau} + \sigma^2_{\alpha\gamma\tau}) + rvy\sum\gamma^2_h + cvy(\sigma^2_\rho + \sigma^2_{\alpha\rho} + \sigma^2_{\gamma\rho} + \sigma^2_{\alpha\gamma\rho} + \sigma^2_{\rho\tau} + \sigma^2_{\gamma\rho\tau}$

$+ \sigma^2_{\alpha\rho\tau} + \sigma^2_\varepsilon).$

The expected value of the sum of squares for years, replicates and varieties is equal to $E[\frac{1}{c}\sum\sum\sum_{gij} X^2_{g\cdot ij}] = \frac{1}{c}\sum\sum\sum_{gij} E[c(\mu + \alpha_g + 0 + 0 + \rho_i + (\alpha\rho)_{gi} + 0 + 0$

$+ \tau_j + (\rho\tau)_{ij} + (\alpha\tau)_{gj} + 0 + 0 + 0 + (\alpha\rho\tau)_{gij}) + \sum_h\varepsilon_{ghij}]^2 = crvy(\mu^2$

$+ \sigma^2_\alpha + \sigma^2_\rho + \sigma^2_{\alpha\rho} + \sigma^2_\tau + \sigma^2_{\rho\tau} + \sigma^2_{\alpha\tau} + \sigma^2_{\alpha\rho\tau}) + rvy\,\sigma^2_\varepsilon.$

The expected value of the sum of squares for cuts, replicates and varieties is equal to $E[\frac{1}{y}\sum\sum\sum_{hij} X^2_{\cdot hij}] = \frac{1}{y}\sum\sum\sum_{hij} E[y\mu + \sum_g \alpha_g + y\,\gamma_h + \sum_g(\alpha\gamma)_{gh} + y\rho_i$

$+ \sum_g(\alpha\rho)_{gi} + y(\gamma\rho)_{hi} + \sum_g(\alpha\gamma\rho)_{ghi} + y\tau_j + y(\rho\tau)_{ij} + \sum_g(\alpha\tau)_{gj}$

$+ y(\gamma\tau)_{hj} + \sum_g(\alpha\gamma\tau)_{ghj} + y(\gamma\rho\tau)_{hij} + \sum_g(\alpha\rho\tau)_{gij} + \sum_g\varepsilon_{ghij}]^2$

$= crvy\,\mu^2 + crv\,\sigma^2_\alpha + rvy\sum\gamma^2_h + crv\,\sigma^2_{\alpha\gamma} + crvy\,\sigma^2_\rho + crv\,\sigma^2_{\alpha\rho}$

$+ crvy\,\sigma^2_{\gamma\rho} + crv\,\sigma^2_{\alpha\gamma\rho} + crvy\,\sigma^2_\tau + crvy\,\sigma^2_{\rho\tau} + crv\,\sigma^2_{\alpha\tau}$

$+ crvy\,\sigma^2_{\gamma\tau} + crv\,\sigma^2_{\alpha\gamma\tau} + crvy\,\sigma^2_{\gamma\rho\tau} + crv\,\sigma^2_{\alpha\rho\tau} + crv\,\sigma^2_\varepsilon.$

The coefficients of the various components are summarized in Table 2. The expectation of the mean squares are presented in Table 3.

Since cuts represent the population, the various effects do not affect the variance of a treatment, year, or replicate (or any combination of these) means, i.e., there is no sampling error associated with obtaining only a sample of cuts rather than all of them. Also as a result of having a finite population of cuts, the coefficient c/c-1 appears in the expectations of the mean squares involved with cuts.

The average genetic progress due to the selections of the largest (or of the largest k) individual (s) in the sample rather than selecting the individual (s) at random from a sample of m lines is

$$\frac{\sigma_{\tau}^{2}\,\bar{x}_{m}}{\sqrt{\dfrac{\sigma_{\varepsilon}^{2}}{yrc} + \dfrac{\sigma_{\alpha\rho\tau}^{2}}{ry} + \dfrac{\sigma_{\alpha\tau}^{2}}{y} + \dfrac{\sigma_{\rho\tau}^{2}}{r} + \sigma_{v}^{2}}}$$

where the estimates of the variance components are used and $\bar{x}_m$ is the expected value of the largest individual (s) from a sample of size m in a normal population with mean zero and unit variance. In making use of the above formula it must be assumed that the effects involving years, replicates and varieties are normally and independently distributed.

Table 2. Coefficients of components for expected value of the various sums of squares.

Sum of squares	components															
	μ^2	σ^2_α	$\sum_h \gamma^2_h$	$\sigma^2_{\alpha\gamma}$	σ^2_ρ	$\sigma^2_{\alpha\rho}$	$\sigma^2_{\gamma\rho}$	$\sigma^2_{\alpha\gamma\rho}$	σ^2_τ	$\sigma^2_{\alpha\tau}$	$\sigma^2_{\gamma\tau}$	$\sigma^2_{\rho\tau}$	$\sigma^2_{\alpha\gamma\tau}$	$\sigma^2_{\alpha\rho\tau}$	$\sigma^2_{\gamma\rho\tau}$	σ^2_ε
$\sum\sum\sum\sum X^2_{ghij}$	rcvy	rcvy	rvy	rcvy	rcvy	rcvy	rcvy	rcvy	rcvy	rcvy	rcvy	rcvy	rcvy	rcvy	rcvy	rcvy
$X^2_{\bullet\bullet\bullet\bullet}/crvy$	rcvy	crv	0	0	cvy	cv	0	0	cry	rc	0	cy	0	c	0	1
$\sum_g X^2_{g\bullet\bullet\bullet}/crv$	rcvy	rcvy	0	0	cvy	cvy	0	0	cry	cry	0	cy	0	cy	0	y
$\sum_h X^2_{\bullet h\bullet\bullet}/rvy$	rcvy	crv	rvy	crv	cvy	cv	cvy	cv	cry	rc	cry	cy	cr	c	cy	c
$\sum\sum_{gh} X^2_{gh\bullet\bullet}/rv$	rcvy	rcvy	rvy	crvy	cvy	cvy	cvy	cvy	rcy	cry	cry	cy	cry	cy	cy	cy
$\sum_i X^2_{\bullet\bullet i\bullet}/cvy$	rcvy	crv	0	0	crvy	crv	0	0	cry	rc	0	cry	0	cr	0	r
$\sum\sum X^2_{g\bullet i\bullet}/cv$	rcvy	crvy	0	0	crvy	crvy	0	0	cry	cry	0	cry	0	cry	0	ry
$\sum\sum X^2_{\bullet hi\bullet}/yv$	rcvy	crv	rvy	crv	crvy	crv	crvy	crv	cry	cr	cry	cry	cr	cr	cry	cr
$\sum\sum\sum X^2_{ghi\bullet}/v$	rcvy	crvy	rvy	crvy	crvy	crvy	crvy	crvy	cry	cry	cry	cry	cry	cry	cry	cry
$\sum X^2_{\bullet\bullet\bullet j}/cry$	rcvy	crv	0	0	cvy	cv	0	0	crvy	crv	0	cvy	0	cv	0	v
$\sum\sum X^2_{g\bullet\bullet j}/cr$	rcvy	crvy	0	0	cvy	cvy	0	0	crvy	crvy	0	cvy	0	cvy	0	vy
$\sum\sum X^2_{\bullet h\bullet j}/ry$	crvy	crv	rvy	crv	cvy	cv	cvy	cv	crvy	crv	crvy	cvy	crv	cv	cvy	cv
$\sum\sum X^2_{\bullet\bullet ij}/cy$	crvy	crv	0	0	crvy	crv	0	0	crvy	crv	0	crvy	0	crv	0	rv
$\sum\sum\sum X^2_{gh\bullet j}/r$	crvy	crvy	rvy	crvy	cvy	cvy	cvy	cvy	crvy	crvy	crvy	cvy	crvy	cvy	cvy	cvy
$\sum\sum\sum X^2_{g\bullet ij}/c$	crvy	crvy	0	0	crvy	crvy	0	0	crvy	crvy	0	crvy	0	crvy	0	rvy
$\sum\sum\sum X^2_{\bullet hij}/y$	crvy	crv	rvy	crv	crvy	crv	crvy	crv	crvy	crv	crvy	crvy	crv	crv	crvy	crv

Table 3. Expectations of mean squares in the analysis of variance for the experiment on perennial crops.

Source of variation	Degrees of freedom	σ^2_{ε}	$\sigma^2_{\alpha\rho\tau}$	$\sigma^2_{\gamma\rho\tau}$	$\sigma^2_{\alpha\gamma\tau}$	$\sigma^2_{\gamma\tau}$	$\sigma^2_{\alpha\tau}$	$\sigma^2_{\rho\tau}$	σ^2_{τ}	$\sigma^2_{\alpha\gamma\rho}$	$\sigma^2_{\gamma\rho}$	$\sigma^2_{\alpha\rho}$	σ^2_{ρ}	$\sigma^2_{\alpha\gamma}$	$\Sigma\gamma^2_h$	σ^2_{α}
Years	y-1	1	c				cr					cv				crv
Cuts	c-1	1		$\frac{cy}{c-1}$	$\frac{cr}{c-1}$	$\frac{cry}{c-1}$				$\frac{cv}{c-1}$	$\frac{cvy}{c-1}$			$\frac{crv}{c-1}$	$\frac{rvy}{c-1}$	
Cut x year	(y-1)(c-1)	1			$\frac{cr}{c-1}$					$\frac{cv}{c-1}$				$\frac{crv}{c-1}$		
Replicates	r-1	1	c					cy				cv	cvy			
Replicate x year	(r-1)(y-1)	1	c									cv				
Replicate x cut	(r-1)(c-1)	1		$\frac{cy}{c-1}$						$\frac{cv}{c-1}$	$\frac{cvy}{c-1}$					
Rep x cut x year	(r-1)(c-1)(y-1)	1								$\frac{cv}{c-1}$						
Varieties	v-1	1	c				cr	cy	cry							
Variety x replicate	(v-1)(r-1)	1	c					cy								
Variety x year	(v-1)(y-1)	1	c				cr									
Variety x cut	(v-1)(c-1)	1		$\frac{cy}{c-1}$	$\frac{cr}{c-1}$	$\frac{cry}{c-1}$										
Variety x year x cut	(v-1)(c-1)(y-1)	1			$\frac{cr}{c-1}$											
Variety x cut x rep	(v-1)(c-1)(r-1)	1		$\frac{cy}{c-1}$												
Variety x year x rep	(v-1)(y-1)(r-1)	1	c													
Residual	(v-1)(y-1)(c-1)(r-1)	1														