REPRESENTATION OF GENERALIZED INVERSE MATRICES

BU-249-M N. S. Urquhart August, 1967

Abstract

The generalized inverse of the matrix A is any matrix G for which AGA = A. Unless A is nonsingular and square, G is not unique. This paper presents two characterizations of the arbitrariness in G: (i) in terms of the relation among elements of G and (ii) in terms of any other generalized inverse.

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1. Introduction

A matrix G which satisfies $AGA = A_{n\times n}$ is called a generalized inverse (g-inverse or g.i.) of A. Of course G is unique when A is square and nonsingular, i.e. $G = A^{-1}$. However, in any other case G is not unique.

For purposes of teaching, computing and research involving g-inverses, it would be helpful to have a characterization of the nature of the arbitrariness in G. A little reflection will show that arbitrary manifests itself in two ways here: (i) in the relation among elements of G and (ii) in the relation of one g-inverse to another. These will be the topics of the subsequent two sections.

2. Relation Among the Elements of G

The equations $AGA = A_{n\times n}$ are consistent for G. The easiest way to show this is to get a solution which satisfies the equations. Assume, without loss of generality, that the first r rows and first r columns of A are linearly independent. This is a theoretical convenience, which, when it comes to computation in practice, may require a computer program to exhibit a set of r linearly independent rows and columns. We will return to this and related computational problems later.

The assumptions allow us to write

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} I \\ A_{21}A_{11}^{-1} \end{bmatrix} A_{11} \left[I,\ A_{11}^{-1}A_{12} \right] = L_1 A_{11} L_2' , \qquad (1)$$

where A_{11} is an $r \times r$ nonsingular matrix.

The equation, $AGA = A$, defining G thus becomes

$$L_1 A_{11} L_2' G L_1 A_{11} L_2' = L_1 A_{11} L_2' \ . \tag{2}$$

Pre- and post-multiplication of (2) by $A_{11}^{-1}(L_1'L_1)^{-1}L_1'$ and $L_2(L_2'L_2)^{-1}A_{11}^{-1}$ respectively yields

$$L_2' G L_1 = A_{11}^{-1} \tag{3}$$

or

$$G_{11} + A_{11}^{-1}A_{12}G_{21} + G_{12}A_{21}A_{11}^{-1} + A_{11}^{-1}A_{12}G_{22}A_{21}A_{11}^{-1} = A_{11}^{-1} \ .$$

Consequently, since G must satisfy <u>only</u> (3), G_{12}, G_{21} and G_{22} are completely arbitrary. However since G_{11} must compensate for whatever values G_{12}, G_{21} and G_{22} have, a general form for G is

$$G = \begin{bmatrix} A_{11}^{-1} - A_{11}^{-1}A_{12}G_{21} - G_{12}A_{21}A_{11}^{-1} - A_{11}^{-1}A_{12}G_{22}A_{21}A_{11}^{-1} & G_{12} \\ & \\ G_{21} & G_{22} \end{bmatrix} \tag{4}$$

Straightforward operations show that G satisfies $AGA = A$ and thus verifies the existence of a solution for G.

Several results follow rather simply from (4). For example, it shows that symmetry of A does not imply symmetry of G; however if A is symmetric and $G_{12} = G_{21}'$ and $G_{22} = G_{22}'$, then G will be symmetric.

Also, results concerning the rank of G can be simply derived from (4). Since $AGA = A$, rank (G) $\geq$ rank (A). A G with this rank is given by taking G_{12}, G_{21}, G_{22} to all be null. On the other hand, $G_{21} = 0$, $G_{22} = I$ (augmented by zeros if necessary) and $G_{12} = -A_{11}^{-1}A_{12}$ gives

$$G = \begin{bmatrix} A_{11}^{-1} & -A_{11}^{-1}A_{12} \\ 0 & I \end{bmatrix} \tag{5}$$

which obviously has rank min (m,n). By choosing G_{22} to be an identity of smaller order augmented by zeros, $G_{12} = -A_{11}^{-1}A_{12}G_{22}$ and $G_{21} = 0$ it follows that G can have any rank between these two limits, i.e. rank (A) ≤ rank (G) ≤ min (m,n) where the equalities are attainable.

The form (4) suggests a computationally easy way to obtain a g-inverse, namely take $G_{12} = 0$, $G_{21} = 0$, $G_{22} = 0$ so that

$$G = \begin{bmatrix} A_{11}^{-1} & 0 \\ 0 & 0 \end{bmatrix} . \tag{6}$$

The form of this result rests very heavily upon having assumed that the first r rows and columns of A were linearly independent. If B is a matrix whose first r rows and columns are not the linearly independent ones, it is possible to find permutation matrices (one element equal to 1 in each row and column and all other elements equal to zero) P_1 and P_2 such that $P_1BP_2 = A$, where A has the required independence property. When G is any g-inverse of A, the matrix

$$P_2GP_1 \tag{7}$$

is a g-inverse of B since $P_1P_1' = P_1'P_1 = I$ (from the definition of a permutation matrix) gives

$$B = P_1'AP_2' \tag{8}$$

and

$$\left.\begin{aligned} BP_2GP_1B &= P_1'AP_2'P_2GP_1P_1'AP_2' \\ &= P_1'AGAP_2' \\ &= P_1'AP_2' = B \ . \end{aligned}\right\} \qquad (9)$$

If we consider G to be given by (6), then the g-inverse of B given by (7) can be computed as follows:

1. Locate the linearly independent rows and columns and denote their subscripts by i_1, i_2, $\cdots$, i_r and j_1, j_2, $\cdots$, j_r.
2. Form the matrix A_{11} where $(A_{11})_{st} = b_{i_s j_t}$ $\quad s,t = 1, 2, \cdots, r$.
3. Invert A_{11} to get $A_{11}^{-1} = (a_{11}^{st})$.
4. Form $(P_2GP_1)_{mn} = \begin{cases} a_{11}^{st} & \text{when } m = j_s,\ n = i_t,\ s,t = 1, 2, \cdots, r \\ 0 & \text{otherwise} \end{cases}$

For example if B is 6 X 7 with rows 1, 2, 4 and columns 1, 3, 5 linearly independent, then

$$\underset{6 \times 6}{P_1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad \underset{7 \times 7}{P_2} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad (10)$$

$$A_{6\times 7} = P_1BP_2 = \begin{bmatrix} b_{11} & b_{13} & b_{15} & b_{12} & b_{14} & b_{16} & b_{17} \\ b_{21} & b_{23} & b_{25} & b_{22} & b_{24} & b_{26} & b_{27} \\ b_{41} & b_{43} & b_{45} & b_{42} & b_{44} & b_{46} & b_{47} \\ b_{31} & b_{33} & b_{35} & b_{32} & b_{34} & b_{36} & b_{37} \\ b_{51} & b_{53} & b_{55} & b_{52} & b_{54} & b_{56} & b_{57} \\ b_{61} & b_{63} & b_{65} & b_{62} & b_{64} & b_{66} & b_{67} \end{bmatrix} \qquad (11)$$

$$P_2GP_1 = P_2 \begin{bmatrix} A_{11}^{-1} & 0 \\ 0 & 0 \end{bmatrix} P_1 = \begin{bmatrix} a_{11}^{11} & a_{11}^{12} & 0 & a_{11}^{13} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ a_{11}^{21} & a_{11}^{22} & 0 & a_{11}^{23} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ a_{11}^{31} & a_{11}^{32} & 0 & a_{11}^{33} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{12}$$

In the event that B is positive semi-definite symmetric as it would be for most statistical applications, the computation of P_2GP_1 can be made substantially simpler. This is because $(i_1, \cdots, i_r)$ can be taken to be the same as $(j_1, \cdots, j_r)$. Consequently $P_2 = P_1'$, i.e. $A = P_1BP_1'$. In this event a regular inversion routine can be modified slightly to yield a g-inverse of B. In a routine which proceeds row-wise, each appearance of a zero on the main diagonal generally corresponds to a dependent row since B is assumed to be psd. Set all of the elements of the corresponding row and column in/the g-inverse equal to zero and ignore this row and column in subsequent calculations. The resulting matrix will be a g-inverse of B. Of course the crux of the computational problem will be the definition of zero.

3. The Relation of One g-inverse to Any Other

Consider G_1 to be any g-inverse of A, i.e. $AG_1A = A$. The matrix

$$G = G_1AG_1 + (I - G_1A)X + Y(I - AG_1) + (I - G_1A)Z(I - AG_1) \;, \tag{13}$$

where X, Y and Z are arbitrary matrices, is a g-inverse of A and as X, Y and Z range over all possible matrices of the correct dimension, every possible g-inverse of A will result. G given by (13) is a g-inverse since

$$AGA = AG_1AG_1A + 0 + 0 + 0 = A \;.$$

As X, Y and Z range over all possible matrices of the correct dimension, they will assume the values $X = G^*$, $Y = G^*$ and $Z = -G^*$ where G^* is any matrix which satisfies $AG^*A = A$. For these matrices, the rhs of (13) is

$$G_1AG_1 + (I - G_1A)X + Y(I - AG_1) + (I - G_1A)Z(I - AG_1)$$

$$= G_1AG_1 + G^* - G_1AG^* + G^* - G^*AG_1 - G^* + G_1AG^* + G^*AG_1 - G_1AG^*AG_1 = G^*. \quad (14)$$

This shows that (13) will yield all possible g-inverses of A.

It is well known that if the equations $A\underline{x} = \underline{y}$ are consistent, then $\underline{x} = G\underline{y} + (I - GA)\underline{z}$, where $\underline{z}$ is arbitrary, is the general solution. A comparison of this result with (13) would suggest that (13) contains too many terms. However a comparison of (13) with (4) will show why this is not the case. Consider

$$G_1 = \begin{bmatrix} A_{11}^{-1} & 0 \\ 0 & 0 \end{bmatrix} . \quad (15)$$

Then the terms of (13), in order, are

$$G_1AG_1 = \begin{bmatrix} A_{11}^{-1} & 0 \\ 0 & 0 \end{bmatrix} , \quad (16)$$

$$(I - G_1A)X = \begin{bmatrix} A_{11}^{-1}A_{12}X_{21} & -A_{11}^{-1}A_{12}X_{22} \\ X_{21} & X_{22} \end{bmatrix} , \quad (17)$$

$$Y(I - AG_1) = \begin{bmatrix} -Y_{12}A_{21}A_{11}^{-1} & Y_{12} \\ -Y_{22}A_{21}A_{11}^{-1} & Y_{22} \end{bmatrix} , \quad (18)$$

and

$$(I - G_1A)Z(I - AG_1) = \begin{bmatrix} A_{11}^{-1}A_{12}Z_{22}A_{21}A_{11}^{-1} & -A_{11}^{-1}A_{12}Z_{22} \\ -Z_{22}A_{21}A_{11}^{-1} & Z_{22} \end{bmatrix} . \qquad (19)$$

Now the comparison with (4) shows that G_1AG_1 introduces A_{11}^{-1} into G_{11} and that each of the subsequent terms above provides for the necessary arbitrariness in G_{21}, G_{12}, G_{22} and compensates for their corresponding effect on G_{11}.

The form (13) probably will not be useful for computational purposes. Rather it will be useful in work where the structure of a generalized inverse is needed.