GEOMETRICAL INTERPRETATION OF STEP-WISE ESTIMATION OF PARAMETERS IN LINEAR REGRESSION ANALYSIS

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ABSTRACT

This note shows the geometrical approach of step-wise estimation of parameters in linear regression analysis given by Freund, Vail and Clunies-Ross in March, 1961 and by Goldberger and Jochems in March, 1961 and by Goldberger in December, 1961 in Journal of American Statistical Association.

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NOTE ON THE GEOMETRICAL INTERPRETATION OF STEP-WISE ESTIMATION OF PARAMETERS IN LINEAR REGRESSION ANALYSIS

Introduction

Least squares estimates of β_1 and β_2 of the partitioned linear model

$$\underset{N\times 1}{Y} = \underset{N\times P_1}{X_1}\ \underset{P_1\times 1}{\beta_1} + \underset{N\times P_2}{X_2}\ \underset{P_2\times 1}{\beta_2} + \underset{N\times 1}{\epsilon_1} \qquad - - - - (0)$$

will be

$$\begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{bmatrix} = \begin{bmatrix} X_1'X_1 & X_1'X_2 \\ X_2'X_1 & X_2'X_2 \end{bmatrix} \begin{bmatrix} X_1'Y \\ X_2'Y \end{bmatrix}$$

$$= \begin{bmatrix} (X_1'X_1)^{-1}[I + X_1'X_2MX_2'X_1(X_1'X_1)^{-1}] & -(X_1'X_1)^{-1}X_1'X_2M^{-1} \\ -M^{-1}X_2'X_1(X_1'X_2)^{-1} & M^{-1} \end{bmatrix} \begin{bmatrix} X_1'Y \\ X_2'Y \end{bmatrix}$$

where $M = X_2'X_2 - X_2'X_1(X_1'X_2)^{-1}X_1'X_2$

$$= \begin{bmatrix} (X_1'X_1)^{-1}X_1'Y - (X_1'X_1)^{-1}X_1'X_2M^{-1}X_2'WY \\ M^{-1}X_2'WY \end{bmatrix} \qquad - - - - (1)$$

where $W = [I - X_1(X_1'X_1)^{-1}X_1]$.

Inserting $M^{-1}X_2'WY = \hat{\beta}_2$ in equation for $\hat{\beta}_1$ we obtain

$$\hat{\beta}_1 = (X_1'X_1)^{-1}X_1'Y - (X_1'X_1)^{-1}X_1'X_2\hat{\beta}_2 \qquad - - - - (2)$$

If one attempts* to estimate β_1 and β_2 in two stages, then as Freund, Vail and Clunies-Ross [1], and Goldberger and Jochems [2] have shown, estimated β_2.

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* For application of this method see Goldberger and Jochems [2].

and related test will be biased. Later Goldberger [3] has unified the discussion for the estimated β_1 and β_2. Before giving the simple geometrical interpretation of their result, we will summarize their estimation results as follow:

One Stage Estimation of β_1 and Two-Stage Estimation of β_2.

First From

$$Y = X_1\beta_1 + \epsilon_2 \qquad \text{- - - - (3)}$$

estimate β_1, call it b_1

$$b_1 = (X_1'X_1)^{-1}X_1'Y \qquad \text{- - - - (4)}$$

Comparison of (4) and (2) results

$$b_1 = \hat{\beta}_1 + (X_1'X_1)^{-1}X_1'X_2\hat{\beta}_2 \qquad \text{- - - - (5)}$$

Compute the estimated residuals $\hat{\epsilon}_2$

$$\begin{aligned} \hat{\epsilon}_2 &= Y - X_1b_1 \\ &= [I - X_1(X_1'X_1)^{-1}X_1']Y \\ &= WY \quad . \qquad \text{- - - - (6)} \end{aligned}$$

Second From

$$\hat{\epsilon}_2 = X_2\beta_2 + \epsilon_3 \qquad \text{- - - - (7)}$$

estimate β_2, call it b_2

$$\begin{aligned} b_2 &= (X_2'X_2)^{-1}X_2'\hat{\epsilon}_2 \\ &= (X_2'X_2)^{-1}X_2'WY \quad . \qquad \text{- - - - (8)} \end{aligned}$$

Inserting $I = MM^{-1}$ in relation (8) we obtain

$$\begin{aligned} b_2 &= (X_2'X_2)^{-1}X_2'MM^{-1}WY \\ &= (X_2'X_2)^{-1}M(M^{-1}X_2'WY) \\ &= (X_2'X_2)^{-1}M\hat{\beta}_2 \qquad \text{by relation (1)} \end{aligned}$$

$$= [I - (X_2'X_2)^{-1}X_2'X_1(X_1'X_1)^{-1}X_1'X_2]\hat{\beta}_2 \quad \text{by definition of } M$$

$$= \hat{\beta}_2 - [(X_2'X_2)^{-1}X_2'X_1(X_1'X_1)^{-1}X_1'X_2]\hat{\beta}_2. \quad ---- (9)$$

It is easy to see from the relation (5) and (9) that

$$Eb_1 \neq E\hat{\beta}_1 = \beta_1$$

$$Eb_2 \neq E\hat{\beta}_2 = \beta_2$$

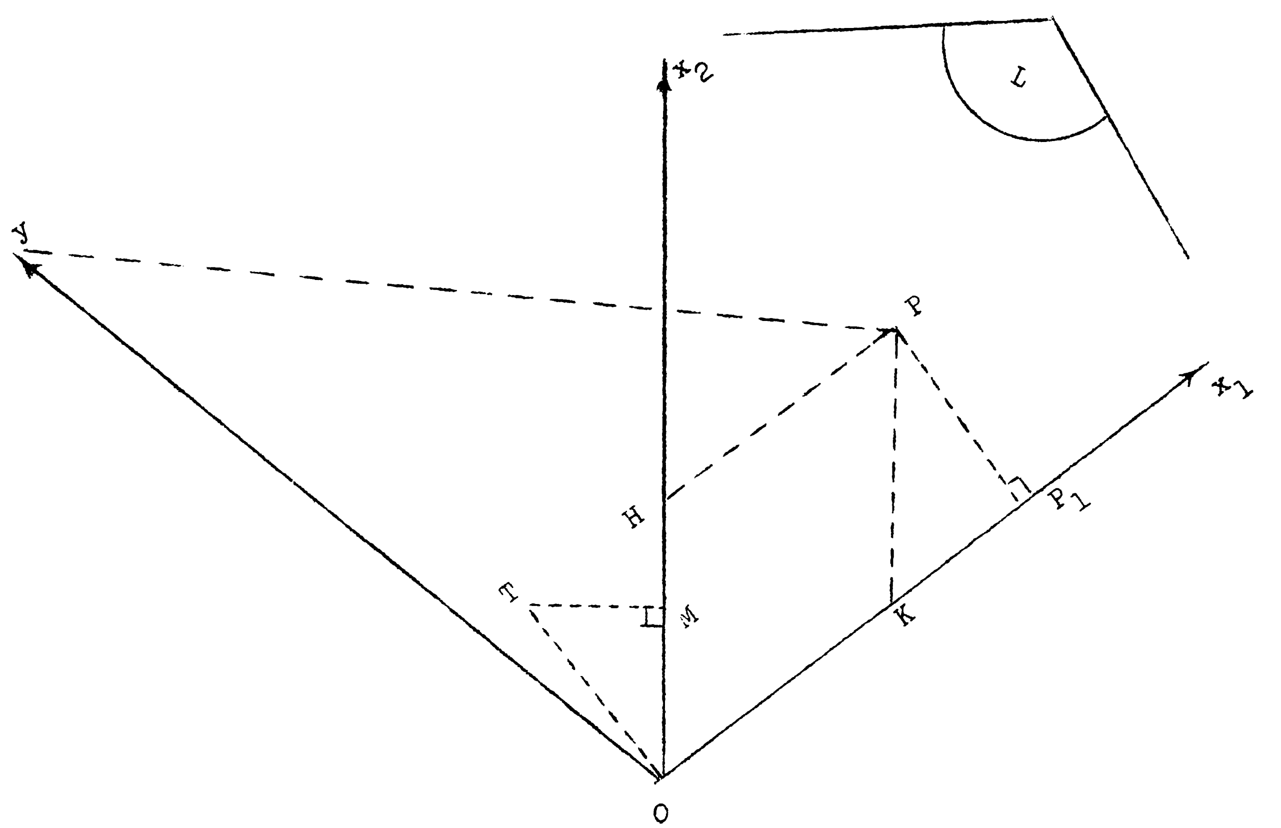
unless $(X_1'X_2) = 0$ or $\beta_2 = 0$.

Geometrical Approach

Consider the case where X_1 and X_2 consist of a single column (the only reason for this choice is so we can provide a diagram). And for convenience we measure Y, X_1, X_2 as deviation from their means. Therefore our model will be

$$y = x_1\beta_1 + x_2\beta_2 + \epsilon.$$

Let L be the linear subspace generated by the column vectors x_1 and x_2. Regression of y on x_1 and x_2 is the orthogonal projection of y on L denoted by P. Coordinates of P gives $\hat{y}$.



Then $\hat{\beta}_1 = \frac{\| OK \|}{\| x_1 \|}$, $\hat{\beta}_2 = \frac{\| OH \|}{\| x_2 \|}$ - - - - (10)

where OK and OH are the components of OP on x_1 and x_2 respectively.

Regression of y on x_1 is the orthogonal projection of y on x_1 or equivalently is the orthogonal projection of $\hat{y}$ on x_1 denoted by P_1. If we complete the parallelogram which has OP as diagonal and OP_1 and PP_1 as sides, we obtain the parallelogram $OTPP_1$. Coordinates of T gives the estimated residuals $\hat{\epsilon}_1$ of regression of y on x_1.

Now the regression of $\hat{\epsilon}_1$ on x_2 is represented by the projection OM of OT on x_2. b_1 and b_2 in relations (5) and (9) for our simple model will be

$$b_1 = \frac{\| OP_1 \|}{\| x_1 \|} , \quad b_2 = \frac{\| OM \|}{\| x_2 \|} \qquad \text{- - - - (11)}$$

Comparing (10) and (11) and by referring to the picture we see that

$$b_1 \neq \beta_1 , \quad b_2 \neq \beta_2 \quad \text{unless} \quad x_1 \perp x_2$$

Remark: There is an unbiased two-stage least squares estimation of parameters. For algebraic derivation see Freund [1] and for the geometric interpretation see pp 112-115 of Applied Regression Analysis by Draper and Smith.

Literature Cited

[1] Freund, R. J., Vail, R. W. and Clunies-Ross, C. W. Residual Analysis. Journal of American Stat. Association 56, 1961.

[2] Goldberger, A. S. and Jochems, D. B. Note on Stepwise Least Squares. Journal of American Stat. Association 56, 1961.

[3] Goldberger, A. S. Stepwise Least Squares: Residual Analysis and Specification Error. Journal of American Stat. Association 56, 1961.

[4] Draper, N. R. and Smith, H. Applied Regression Analysis. John Wiley and Son, Inc., New York, 1966.