The Geometric Interpretation of Missing Observations

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Abstract

This paper presents the geometric interpretation of missing observations in the general linear model.

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The observation space V is decomposed into $V_1 \oplus V_2$, $V_1 \perp V_2$. We only observe $\mathcal{Y}' \in V_1$. V_2 is the space of missing observations.

Let Ω be the original mean space, with $\Omega \not\subset V_1$, $\Omega \not\subset V_2$, $\Omega \neq V$. If $\mu \in \Omega$, $\mathcal{Y}'$ has expectation $P_{V_1} \mu = \mu_1$, which runs around in $\Omega_1 = P_{V_1} \Omega$, the mean space corresponding to those observations observed. Denote $P_{V_2} \Omega = \Omega_2$. We seek $P_{\Omega_1} \mathcal{Y}'$, the b.l.u.e. of $E\, \mathcal{Y}'$.

Now we could compute $P_{\Omega_1} \mathcal{Y}'$ directly, but the missing observation method utilizes the fact that P_Ω, P_{V_1}, and P_{V_2} are known explicitly.

We want an easy way of computing a point $\xi \in V_2$ (the missing observations) s.t. $P_{V_1} P_\Omega (\xi + \mathcal{Y}') = P_{\Omega_1} \mathcal{Y}'$, for all $\mathcal{Y}' \in V_1$. Thus, by completing the missing observations with ξ, applying the standard analysis and then projecting the least squares estimate onto V_1, we get the least squares estimate of $\mu_1 = E\, \mathcal{Y}'$, based on the observed $\mathcal{Y}'$.

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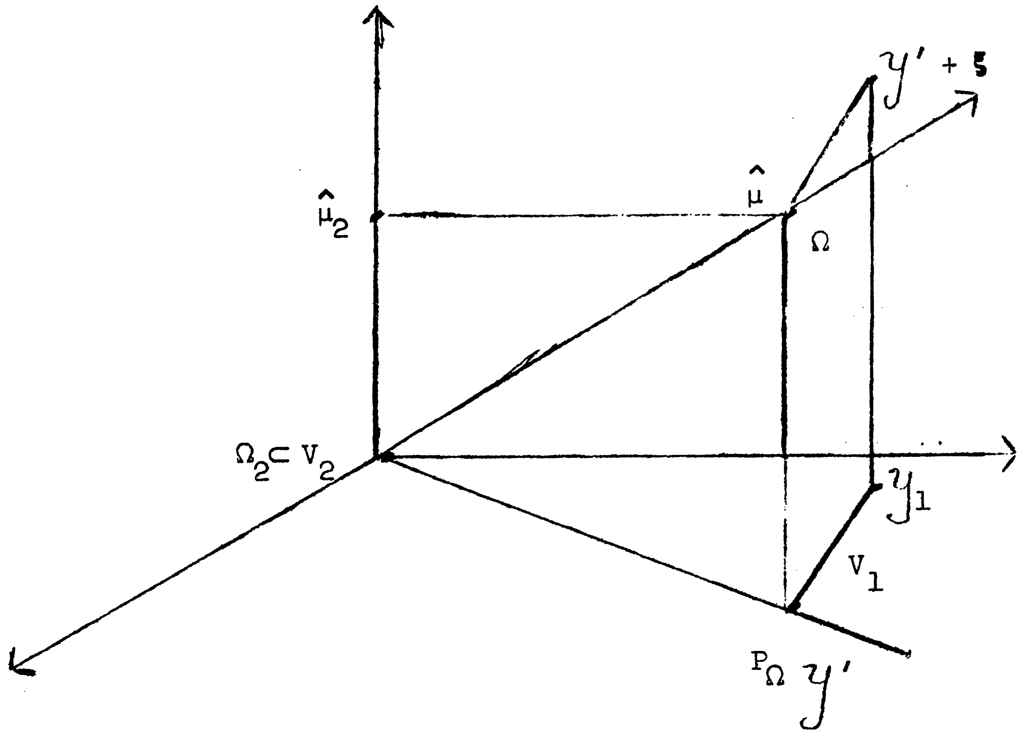

<u>Lemma</u>.

Let $L(y')$ be the space of solutions $z \in V_2$ of the equation

$$P_{V_2} P_\Omega (y' + z) = z, \text{ for each } y' \in V_1.$$

Let $L^*(y')$ be the space of solns. $z \in V_2$ to the equation

$$P_{V_1} P_\Omega (y' + z) = P_{\Omega_1} y'.$$

Then, for all $y' \in V_1$, $L(y') \neq \phi$, and $L(y') = L^*(y')$.

If $\dim \Omega_1 < \dim \Omega$, $L(y')$ has dim h, $h = \dim \Omega - \dim \Omega_1$.

Proof:

(i) $z, z' \in L(\mathcal{Y}') \Rightarrow z - z' = P_{V_2} P_{\Omega} (z - z')$.

By Bessel's inequality,

$\|z - z'\| \geq \|P_{V_2} P_{\Omega} (z - z')\|$, with equality holding iff $P_{V_2} P_{\Omega}$ operates as the identity map on $z - z'$. Hence, $z - z' \in \Omega \cap V_2$.

(ii) let $z, z' \in L^*(\mathcal{Y}')$. Then

$$0 = P_{V_1} P_{\Omega} (\mathcal{Y}' + z) - P_{V_1} P_{\Omega} (\mathcal{Y}' + z')$$
$$= P_{V_1} P_{\Omega} (z - z').$$ So $z - z' \in \text{Ker } P_{V_1} P_{\Omega}$.

But $\text{Ker } P_{V_1} = V_2$. So $\text{Ker } P_{V_1} P_{\Omega} = V_2 \cap \Omega$.

(iii) So $\dim L^*(\mathcal{Y}') = \dim L(\mathcal{Y}')$.

(iv) For each $\mu_1 \in \Omega_1$, $\mu_1 \neq 0$, let $S(\mu_1) =$

$P_{V_1}^{-1} \{\mu_1\} \cap \Omega$. $S(\mu_1) \neq \phi$, $\forall\ \mu_1 \neq 0$.

$\therefore$ For each $\mathcal{Y}' \in V_1$, $\exists\ \hat{\mu} \neq 0$ s.t. $\hat{\mu} \in S(P_{\Omega_1} \mathcal{Y}')$.
But $\hat{\mu} = P_{V_1} \hat{\mu} + P_{V_2} \hat{\mu} = P_{\Omega_1} \mathcal{Y}' + \hat{\mu}_2$, say.

Now $\hat{\mu} = P_{\Omega}\,(\hat{\mu}) = P_{\Omega}\,(P_{\Omega_1} y' + \hat{\mu}_2)$

$= P_{\Omega}\,(P_{\Omega_1} y' - y') + P_{\Omega}\,(y' + \hat{\mu}_2)$

$= P_{\Omega}\,(y' + \hat{\mu}_2)$, for $P_{\Omega_1} y' - y' + \Omega_1$ and Ω_2,

and $\Omega \subset \Omega_1 \oplus \Omega_2$, so $P_{\Omega_1} y' - y' + \Omega$.

Thus, $\hat{\mu}_2 = P_{V_2}\,\hat{\mu} = P_{V_2}\,P_{\Omega}\,(y' + \hat{\mu}_2)$.

So $\hat{\mu}_2 \in L(y')$.

(v) Further, $P_{V_1}\,P_{\Omega}\,(y' + \hat{\mu}_2) = P_{V_1}\,\hat{\mu} = P_{\Omega_1} \quad '$.

So $\hat{\mu}_2 \in L^*(y')$.

Thus, we see that $L(y') = L^*(y') =$ the coset $\hat{\mu}_2 + \Omega \cap V_2$.

(vi) Now, $\Omega \cap V_2 = \operatorname{Ker} P_{V_1}\,P_{\Omega}$

$\dim \operatorname{Ker} P_{V_1} \mid \Omega = \dim \Omega - \dim \operatorname{Im} P_{V_1} \mid \Omega$

$= \dim \Omega - \dim \Omega_1$.

Thus, any solution in $L(y')$ will serve equally well to find $P_{\Omega_1} y'$. This generalizes the result of Kruskal [1], where it is assumed that $\dim \Omega = \dim \Omega_1$.

Example:

$$E\, y_{ij} = \alpha_i + \beta_j \qquad i = 1, \cdots, I;\ j = 1, \cdots, J$$

Suppose the i_0, j_0th observation is missing. Then we seek a no. z s.t.

$$\frac{z + y_{\cdot j_0}}{I} + \frac{z + y_{i_0 \cdot}}{J} - \frac{z + y_{\cdot\cdot}}{IJ} = z.$$

Thus,

$$z = \left(\frac{y_{i_0 \cdot}}{J} + \frac{y_{\cdot j_0}}{I} - \frac{y_{\cdot\cdot}}{IJ}\right)\left(\frac{I}{I-1}\right)\left(\frac{J}{J-1}\right).$$

References

[1] William Kruskal, "The Coordinate-Free Approach to Gauss-Markov Estimation, and its Applications to Missing and Extra Observations", Fourth Berkeley Symposium on Math. Stat. and Prob., Vol. 1, pp. 435 - 51.