Steady-State Distribution of Waiting Time in the Queue M/M/1

BU-217-M D. S. Robson May, 1966

ABSTRACT

The asymptotic districution of waiting time in the single-server, "first come, first served" queueing system with Poisson input and exponentially distributed service time,

$$P(W \leq w) = 1 - \frac{\lambda}{\mu} e^{-(\mu-\lambda)w} \quad , \quad 0 \leq w < \infty$$

is derived from the relation

$$P(W \leq w) = P(W + S < X + w)$$

where S and X, representing service time and inter-arrival time, are exponentially distributed with parameters μ and λ, respectively.

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In the single server queueing system M/M/1 with Poisson input, exponentially distributed service time and a "first come, first served" discipline the asymptotic distribution of waiting time is

$$F_W(0) = \frac{\mu - \lambda}{\mu}$$

$$dF_W(w) = \frac{\lambda}{\mu} (\mu - \lambda)e^{-(\mu-\lambda)w}dw, \quad 0 < w < \infty$$

where λ^{-1} is the mean inter-arrival time and μ^{-1} is the mean service time. A derivation of this well known result is given below.

Let W_0 denote the waiting time of a particular customer and let $W_{-1} + S_{-1}$ denote the waiting time plus service time of the preceding customer. If X is the time between the arrivals of these two customers then $W_0 \le w$ if and only if $W_{-1} + S_{-1} \le X + w$; that is,

$$F_{W_0}(w) = \int_0^\infty f_X(x) \int_0^{x+w} f_{S_{-1}}(s)\, F_{W_{-1}}(x + w - s)dsdx \; .$$

In the steady state

$$F_{W_0}(w) = F_{W_{-1}}(w)$$

and

$$f_X(x) = \lambda e^{-\lambda x}$$

$$f_{S_{-1}}(s) = \mu e^{-\mu s} \quad .$$

Hence,

$$F_W(w) = \int_0^{\infty} \lambda e^{-\lambda x} \int_0^{x+w} \mu e^{-\mu s} F_W(x + w - s)dsdw$$

or, letting $z = x + w - s$,

$$F_W(w) = \int_0^{\infty} \lambda e^{-\lambda x} \int_0^{x+w} \mu e^{-\mu(x+w-z)} F_W(z)dzdx$$

with derivative

$$f_W(w) = \int_0^{\infty} \lambda e^{-\lambda x} \left[\mu e^{-\mu(x+w)} F_W(0) + \int_0^{x+w} \mu e^{-\mu(x+w-z)} F_W(z)dz\right] dx$$

$$= \frac{\lambda\mu}{\lambda + \mu} e^{-\mu w} F_W(0) + \int_0^{\infty} \lambda e^{-\lambda x} \int_0^{x+w} \mu e^{-\mu(x+w-z)} F_W(z)dzdx$$

Interchanging the order of integration gives

$$F_W(w) = \lambda\mu e^{-\mu w} \left[\int_0^{w} \int_0^{\infty} e^{\mu z} F_W(z)e^{-(\mu+\lambda)x}dxdz \right.$$

$$\left. + \int_w^{\infty} \int_{z-w}^{\infty} e^{\mu z} F_W(z)e^{-(\mu+\lambda)x}dxdz \right]$$

$$= \frac{\lambda\mu}{\lambda + \mu} e^{-\mu w} \left[\int_0^{w} e^{\mu z}F_W(z)dz + e^{(\mu+\lambda)w} \int_w^{\infty} e^{-\lambda z}F_W(z)dz \right]$$

where, in particular,

$$F_W(0) = \frac{\lambda\mu}{\lambda + \mu} \int_0^{\infty} e^{-\lambda z} F_W(z)dz$$

Similarly,

$$f_W(w) = \frac{\lambda\mu}{\lambda + \mu}\left[e^{-\mu w} F_W(0) + e^{-\mu w}\int_0^w e^{\mu z} f_W(z)dz + e^{\lambda w}\int_w^\infty e^{-\lambda z} f_W(z)dz\right]$$

The characteristic function of f_W is therefore

$$\varphi_+(t) = \int_0^\infty e^{itw} f_W(w)dw$$

$$= \frac{\lambda\mu}{\lambda + \mu}\left[\frac{F_W(0)}{\mu - it} + \int_0^\infty e^{-(\mu-it)w}\int_0^w e^{\mu z} f_W(z)dzdw\right.$$

$$\left. + \int_0^\infty e^{(\lambda+it)w}\int_w^\infty e^{-\lambda z} f_W(z)dzdw\right]$$

Interchanging the order of integration gives

$$\varphi_+(t) = \frac{\lambda\mu}{\lambda + \mu}\left[\frac{F_W(0)}{\mu - it} + \int_0^\infty\int_z^\infty e^{\mu z} f_W(z)e^{-(\mu-it)w}\, dwdz\right.$$

$$\left. + \int_0^\infty\int_0^z e^{-\lambda z} f_W(z)e^{(\lambda+it)w}\, dwdz\right]$$

$$= \frac{\lambda\mu}{\lambda + \mu}\left[\frac{F_W(0)}{\mu - it} + \frac{\varphi_+(t)}{\mu - it} + \frac{\varphi_+(t)}{\lambda + it} - \frac{1}{\lambda + it}\int_0^\infty e^{-\lambda z} f_W(z)dz\right]$$

where

$$\int_0^\infty e^{-\lambda z} f_W(z)dz = e^{-\lambda z} F_W(z)\Big|_0^\infty + \lambda\int_0^\infty e^{-\lambda z} F_W(z)dz$$

$$= -F_W(0) + \frac{\lambda + \mu}{\mu} F_W(0)$$

$$= \frac{\lambda}{\mu} F_W(0)\ .$$

Thus,

$$\varphi_+(t) = \frac{\lambda\mu}{\lambda + \mu}\left[\frac{(\lambda + \mu)it}{\mu(\mu - it)(\lambda + it)} F_W(0) + \frac{\lambda + \mu}{(\mu - it)(\lambda + it)} \varphi_+(t)\right]$$

or

$$\varphi_+(t) = \frac{\lambda F_W(0)}{\mu - \lambda - it} .$$

It then follows that

$$f_W(w) = \lambda F_W(0) e^{-(\mu-\lambda)w}, \quad 0 < w < \infty$$

where

$$F_W(0) = 1 - \int_0^\infty f_W(w)dw$$

$$= 1 - \frac{\lambda}{\mu - \lambda} F_W(0)$$

or

$$F_W(0) = \frac{\mu - \lambda}{\mu} .$$