BU-180-M

SOME REMARKS ON THE GENERALIZED CONSTRUCTION AND ANALYSIS OF FRACTIONAL REPLICATES

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1. Introduction

Consider the s^m factorial (s is a positive prime or power of a positive prime and m any positive integer ≥ 2), then we know that:

(1.1) The s^m treatment combinations are in 1:1 correspondence with the points of the finite Euclidean geometry EG(m,s).

(1.2) The $\frac{s^m-1}{s-1}$ effects are in 1:1 correspondence with the points of the finite Projective geometry PG(m-1,s).

(1.3) $X\beta=Y$, is the set of normal equations for a full replicate; X is an orthogonal matrix in the sense that X'X is diagonal, β is the vector of parameters with μ as its first element and Y is the observation vector.

(1.4) $X_i\beta=Y_i$ is the equation system corresponding to the i^{th} observation Y_i, i.e. X_i is the <u>i^{th} row vector</u> of X corresponding to Y_i, $i=1,2,\cdots,s^m$. This equation follows from (1.3)

(1.5) $X_i'X_i\beta=X_i'Y_i=Z_i$, say; this system is the result of premultiplying (1.4) by X_i'. The result is that Z_i now corresponds to the square matrix $X_i'X_i$, which has rank one.

(1.6) $X_R\beta=Y_R$, is the set of normal equations corresponding to the fraction Y_R. This system can be read off from (1.3).

(1.7) $[X_{R1}\vdots X_{R2}]\begin{bmatrix}\beta_R\\ \beta_0\end{bmatrix} = Y_R$, is the partitioned system, where β_R (with μ as its first element) is selected such that X_{R1} is invertible.

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(1.8) $\beta_R + X_{R1}^{-1}X_{R2}\beta_0 = X_{R1}^{-1}Y_R$ is a solution to the system under (1.7).

(1.9) $\beta_R + (X_{R1}'X_{R1})^{-1}X_{R1}'X_{R2}\beta_0 = (X_{R1}'X_{R1})^{-1}Y_R$ is equivalent to (1.8) but here we essentially select the invertible matrix $X_{R1}'X_{R1}$ from $X_R'X_R$.

2. Definitions

In sections 3 and 4 we will utilize the following definitions:

(2.1) Define the first element of $\beta_R+X_{R1}^{-1}X_{R2}\beta_0$ [or equivalently of $\beta_R + (X_{R1}'X_{R1})^{-1}X_{R1}'X_{R2}\beta_0$] as a defining contrast of the fraction Y_R.

(2.2) Define the whole vector $\beta_R + X_{R1}^{-1}X_{R2}\beta_0$ [or equivalently the vector $\beta_R + (X_{R1}'X_{R1})^{-1}X_{R1}'X_{R2}\beta_0$] as the aliasing scheme or structure of the fraction Y_R.

(2.3) Define Y_R to be a regular fraction if the corresponding treatment combinations satisfy a combination of levels of the generators of a particular r-flat of PG(m-1,s), $0 \le r \le m-2$. [Or simply: Y_R is regular if it belongs to a particular r-flat.] A fraction is irregular if it is not regular.

Example: 2^3 fractional; $Y_R = \{000,111\}$ satisfies the combination (0,0) of the generators (AB,AC) of the 1-flat {AB,AC,BC}. Hence {000,111} is regular.

3. Some general consequences

From sections 1 and 2 one can easily establish the following results:

(3.1) Regular fractions are always of type $\frac{1}{s^n}$, $1 \le n \le m-1$. Hence one can always choose a regular fraction of this type.

(3.2) The defining contrast of a fraction is unique if and only if the fraction is regular.

(3.3) A $\frac{1}{s^n}$ fraction is regular if and only if $X_{R1}^{-1}X_{R2}$ can be put in the form $I_{s^{m-n}} \otimes [\delta\ \delta\ \cdots\ \delta]_{s^n-1}$, where δ is $\pm$ 1.

(3.4) Consider the full replicate with the system $X\beta=Y$ and denote by $X'X$ = Diagonal $(d_1\ d_2\ \cdots\ d_{s^m})$. Let $S_i = \frac{1}{d_i} X_i'X_i = \frac{1}{d_i}$ (i^{th} row of $X)'$(i^{th} row of X), then:

a) S_i is symmetric and idempotent, $i=1,2,\cdots,s^m$,

b) $S_iS_j=0$ for $i \neq j$,

c) $\sum_{j\epsilon J} S_j$ is idempotent for any subset indexed by J,

d) $\sum_{i=1}^{s^m} S_i=I$,

if and only if s=2.

These conclusions arise from the fact that for the 2^n series $X'X=XX'=2^mI$. From this result and equations (1.4), (1.5) and (1.9) it follows that fractional replication for the 2^m series can be studied from the viewpoint of idempotent matrices.

4. Some results concerning 2^m factorials

For the 2^m series it is well known that:

(4.1) $X\beta=Y$, where X is the Hadamard matrix $H_{2^m} = H_2 \otimes H_2 \otimes \cdots \otimes H_2$ with $H_2 = \begin{bmatrix} + & - \\ + & + \end{bmatrix}$.

The following results can be established easily:

(4.2) A fraction consisting of 2 observations is always regular.

(4.3) For a regular $\frac{1}{2^n}$ fraction of a 2^m factorial the solution is

$$\beta_R + I_{2^{m-n}} \otimes [\delta\ \delta\ \cdots\ \delta]_{2^n-1}\beta_0 = \frac{1}{2^{m-n}} G_{2^{m-n}} Y_R, \text{ where } \delta = \pm 1$$

and G is a Hadamard matrix. The covariance matrix is obviously $\frac{\sigma^2}{2^{m-n}} I_{2^{m-n}}$.

(4.4) For irregular fractions of type $\frac{1}{2^n}$ it is always possible to choose X_{R1} in $[X_{R1} \vdots X_{R2}] \begin{bmatrix} \beta_R \\ \beta_0 \end{bmatrix} = Y_R$ as a Hadamard matrix. The covariance matrix in this instance is also $\frac{\sigma^2}{2^{m-n}} I_{2^{m-n}}$.

(4.5) When irregular fractions of t points are taken from the first half of a 2^m factorial, i.e. from $\{000\cdots0, 100\cdots0, 010\cdots0, \cdots, 11\cdots10\}$ then the number of ways of selecting β_R (with μ as its first element) such that $|X_{R1}| \neq 0$ is given by $t/2^m \binom{2^{m-1}}{t} 2^t$, for $3 \leq t \leq 2^{m-1}$. This result also holds when the fraction is taken from the other half. Note that knowledge of the number of β_R's such that $|X_{R1}| \neq 0$ is of importance only for the irregular case, since the solution for the regular case is invariant under such a choice of β_R.

5. Some examples

(5.1) Consider the $\frac{1}{2}$ fraction consisting of $\{000, 100, 010, 110\}$ of the 2^3 factorial. This is a regular fraction, since it satisfies the 0^{th} level of C. The solution is:

$$\begin{bmatrix} \mu - C/2 \\ A/2 - AC/2 \\ B/2 - BC/2 \\ AB/2 - ABC/2 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} + & + & + & + \\ - & + & - & + \\ - & - & + & + \\ + & - & - & + \end{bmatrix} = \frac{1}{4} H_2^{\prime 2} Y_R$$

Defining contrast: $\mu - c/2$

Covariance matrix: $\frac{\sigma^2}{4} I_4$

(5.2) Next consider the irregular $\frac{1}{2}$ fraction {000,100,010,001} and let us choose X_{R1} according to (4.4) then the solution is:

$$\begin{bmatrix} \mu - A/4 - B/4 - C/4 + ABC/4 \\ AB/2 - A/4 - B/4 + C/4 - ABC/4 \\ AC/2 - A/4 + B/4 - C/4 - ABC/4 \\ BC/2 + A/4 - B/4 - C/4 - ABC/4 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} Y_{000} \\ Y_{100} \\ Y_{010} \\ Y_{001} \end{bmatrix} = \frac{1}{4} G_{2^2} Y_R$$

Defining contrast: $\mu - A/4 - B/4 - C/4 + ABC/4$. The

Covariance matrix is also $\frac{\sigma^2}{4} I_4$, since G is a Hadamard matrix.

(5.3) Consider the regular $\frac{1}{3}$ fraction of the 3^2 factorial consisting of {00,01,02}. The solution corresponding to this fraction is:

$$\begin{bmatrix} \mu - A_L + A_Q \\ B_L - A_L B_L + A_Q B_L \\ B_Q - A_L B_Q + A_Q B_Q \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 2 & 2 & 2 \\ -3 & 0 & 3 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} Y_{00} \\ Y_{01} \\ Y_{02} \end{bmatrix}$$

Defining contrast: $\mu - A_L + A_Q$

Covariance matrix: $\frac{\sigma^2}{6} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(5.4) Finally take the irregular $\frac{1}{3}$ fraction {00,01,10} of the 3^2 factorial. A solution is:

$$\begin{bmatrix} \mu - 2A_Q - 2B_Q - A_L B_L + 3A_L B_Q + 3A_Q B_L - 5A_Q B_Q \\ A_L - 3A_Q - A_L B_L + A_L B_Q + 3A_Q B_L - 3A_Q B_Q \\ B_L - 3B_Q - A_L B_L + 3A_L B_Q + A_Q B_L - 3A_Q B_Q \end{bmatrix} = \begin{bmatrix} -1 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} Y_{00} \\ Y_{01} \\ Y_{10} \end{bmatrix}$$

Defining contrast: $\mu - 2A_Q - 2B_Q - A_L B_L + 3A_L B_Q + 3A_Q B_L - 5A_Q B_Q$.

Covariance matrix: $\sigma^2 \begin{bmatrix} 3 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}$.

6. Future research

(6.1) Relationship between the $X_R'X_R$ matrices and selection of β_R such that $|X_{R1}'X_{R1}| \neq 0$. This will be especially important for the 2^m case, since we can work with the algebra of idempotent matrices.

(6.2) Determination of all possible β_R's such that $|X_{R1}| \neq 0$ for any fixed fraction of t observations.

(6.3) Development of a rule to generate the aliasing structures from defining contrasts for irregular fractions.