ON THE INADMISSIBILITY OF NON-NEGATIVE MAXIMUM LIKELIHOOD ESTIMATORS OF THE "BETWEEN-GROUPS" VARIANCE COMPONENT*

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## Abstract


For the balanced one-way classification the maximum likelihood estimators of the "between-groups" variance component $\sigma^2$ take the form

$$\hat{\sigma}^2_c = \begin{cases} 0 & \text{if } B < W \\ \frac{B-cW}{rc} & \text{if } B \geq W \end{cases}$$

where B and W are the "between-groups" and "within-groups" mean squares, respectively, in a one-way array of g groups with r observations each. Alternative values for c have been previously given as c=1 or c=g/(g-1). For the case g=3 a value of c ≥ 2 is required to minimize $E(\hat{\sigma}^2_c-\sigma^2)^2$, and

$$E(\hat{\sigma}^2_2-\sigma^2)^2 < E(\hat{\sigma}^2_{3/2}-\sigma^2)^2 < E(\hat{\sigma}^2_1-\sigma^2) \quad .$$



* Biometrics Unit, Cornell University, Ithaca, New York

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## Introduction and Summary

Herbach (1959) and Thompson (1962) have derived maximum likelihood estimators of the variance components in Eisenhart's Model II (1947) for the balanced one-way classification. The form of the "between-groups" variance component estimator in both cases is:

$$\hat{\sigma}_c^2 = \begin{cases} 0 & \text{for } B < cW \\ \frac{B-cW}{rc} & \text{for } B \geq cW \end{cases}$$

where B and W are the "between-groups" and "within-groups" mean squares, respectively, in a one-way classification with g groups of r observations each. Herbach gives $c=g/(g-1)$ as the maximum likelihood form and Thompson gives $c=1$ as the form for a conditional maximum likelihood estimator.

Here we shall examine the easily manipulated case where $g=3$ and show that $c=2$ gives a uniformly smaller mean squared error than either of the above estimators. In fact, for $g=3$,

$$E(\hat{\sigma}_2^2-\sigma^2)^2 < E(\hat{\sigma}_{3/2}^2-\sigma^2)^2 < E(\hat{\sigma}_1^2-\sigma^2) \quad .$$

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## The Case of g=3 Groups

When g=3 the "between-groups" mean square is exponentially distributed,

$$P(B \geq b) = e^{-b/\theta}$$

where $\theta = \omega^2 + r\sigma^2$ is the expected value of this mean square. The estimator $\hat{\sigma}^2_c$ then has a mean value of

$$E\hat{\sigma}^2_c = E\,\frac{\theta}{rc}\,e^{-cW/\theta} = \frac{\theta}{rc}\left(1 + \frac{2c\omega^2}{3(r-1)\theta}\right)^{-\frac{3(r-1)}{2}}$$

and a mean squared error of

$$E(\hat{\sigma}^2_c - \sigma^2)^2 = \sigma^4 + \frac{2\theta}{rc}\left(\frac{\theta}{rc} - \sigma^2\right)\left(1 + \frac{2c\omega^2}{3(r-1)\theta}\right)^{-\frac{3(r-1)}{2}}$$

If r is large this expression is closely approximated by

$$E(\hat{\sigma}^2_c - \sigma^2)^2 \approx \sigma^4 + \frac{2\theta}{rc}\left(\frac{\theta}{rc} - \sigma^2\right) e^{-c\omega^2/\theta}$$

which attains its minimum value at

$$c = \frac{\omega^4 - r^2\sigma^4 + \theta\sqrt{\theta^2 + 4r\omega^2\sigma^2}}{2r\omega^2\sigma^2} \geq 2 \quad .$$

Since the factor

$$\left(1 + \frac{2c\omega^2}{3(r-1)\theta}\right)^{-\frac{3(r-1)}{2}} \approx e^{-c\omega^2/\theta}$$

is a decreasing function of c, it follows that

$$E(\hat{\sigma}_2^2-\sigma^2)^2 < E(\hat{\sigma}_{3/2}^2-\sigma^2)^2 < E(\hat{\sigma}_1^2-\sigma^2)^2$$

for all $\sigma^2 \geq 0$.

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