APPLICATION OF CYCLIC COLLINEATIONS TO THE CONSTRUCTION OF BALANCED L-RESTRICTIONAL LATTICES

BU-172-M B. L. Raktoe November, 1964

ABSTRACT

The construction of balanced ℓ-restrictional prime powered lattice designs by means of cyclic collineations is shown by limiting the discussion to a 2-restrictional lattice design (or lattice rectangle). Since collineations are defined on the finite projective geometry PG(m-1,s), the 1:1 correspondence between the pseudo effects and the points of PG(m-1,s) is illustrated and confounding schemes in terms of pseudo effects are expressed geometrically. From this last fact it is possible to use the columns of the respective $\frac{s^m-1}{s-1}$ powers of a cyclic collineation to read off the confounding schemes associated with a balanced lattice rectangle. Also an ordered balanced set is given which enables the experimenter to choose any desired best q replicates.

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1. Lattice rectangle

Consider an experiment with 8 treatments in a (4 rows) $\times$ (2 columns) design. If the treatments are identified with the 2^3 combinations of 3 factors each at 2 levels and rows and columns with 2 restrictions, then we speak in this case of a 2-restrictional lattice or lattice rectangle. So far we have not said anything about the number of replicates used.

2. Balanced lattice rectangle

It can be shown that when we have a 2-restrictional lattice design $s^m = s^r \cdot s^c$, ($r+c=m$, s a prime or a power of a prime) then the number of arrangements required for "balancing is given by $\frac{s^m-1}{s-1}$. In our example we need $\frac{2^3-1}{2-1} = 7$ arrangements to get a "balanced" lattice rectangle. The term "balanced" is used in the sense that each pair of treatment combinations occurs together an equal number of times in rows and an equal number of times in columns. This concept is an extension of that used in Balanced Incomplete Block designs. However, there is another interpretation of the term balanced lattice rectangle. Before we give this let us make some ideas more precise.

3. Finite geometries and lattices

The $2^3(=s^m)$ treatment combinations together form what is called a 3(=m)-dimensional finite Euclidean geometry EG(3,2)(=EG(m,s)) based on the Galois field (GF(2)(=GF(s)); that is our field consists, in the case s is a prime, of the residue classes modulo s and, if s is a power of a prime, of polynomial resudüe classes modulo s and modulo P(x). In the 2^3 case above GF(2) consists of the residue classes {0} and {1}. The points of EG(3,2) are: (000), (100), (010), (001), (110), (101), (011), (111).

Also the $\frac{2^3-1}{2-1}\left(=\frac{s^m-1}{s-1}\right)$ effects A, B, C, AB, AC, BC, ABC can be made to correspond to the points of another geometry called PG(2,2) (=PG(m-1,s)) which has as points (100), (010), (001), (110), (101), (011), (111). In this geometry we have done away with the parallel postulate of the Euclidean geometry. There is, however, the following relationship between the two geometries. If we fix the first coordinate of all the points of a PG(m-1,s) we will get an EG(m-1,s) in homogeneous coordinates. Also an effect, say A, is represented by 1(=s-1) degree of freedom in a 2^3 factorial, that is we get this single d.f. by comparison (usually orthogonal contrast) of the levels of A. But the levels of A, which is a point in PG(2,2), are parallel planes of the EG(3,2), i.e. the following 2 planes $\{x_1=0\}$ = {(000),(001),(010),(011)} and $\{x_1=1\}$ = {(100),(101),(110),(111)}. Now since EG(3,2) $\subset$ PG(3,2) we could have used a PG(3,2) with first coordinates fixed and get those levels of A as planes meeting at the line at infinity. This interplay of the two geometries is important in the explanation of lattice or factorial designs.

When we define a balanced lattice rectangle we have a choice in the two geometries. If balancing is defined in terms of treatments then the ideas of the Euclidean geometry EG(m,s) can be used or since this is contained in PG(m+1,s) we can utilize results of this latter geometry. When we define balancing in terms of effects, i.e. a balanced lattice rectangle is a minimal set of confounding schemes such that each effect is confounded in an equal number of schemes with rows and an equal number of schemes with columns, then all the properties of the finite projective geometry PG(m-1,s) can immediately be used to find properties concerning lattice rectangles. It is this latter one we shall be concerned with in the following.

4. PG(m-1,s), Confounding and Balanced lattice rectangle

In our example of the $4\times 2 = 2^2\times 2$ lattice rectangle we know that we must confound $3(=\frac{s^2-1}{s-1})$ effects with rows and $1(=\frac{s^c-1}{s-1})$ effect with columns. But also it is well known that for any row confounding we need only 2(=r) generators to obtain 3 confounded effects and similarly we need consider only 1(=c) generator to get the column confounding. Also it is impossible to confound the same effect in rows as well as in columns, that is, we confound effects in columns different from that in rows. Let us now translate all this in terms of our projective geometry PG(m-1,s).

In PG(m-1,s) any two independent points determine a line (=1-flat), any three independent points determine a plane (=2-flat) and so on, any k independent points determine a (k-1)-flat. Any confounding scheme then corresponds to choosing a line (=(r-1)-flat) of PG(2,2) for the row confounding and a point (=(c-1)-flat), independent of the line for the column confounding. A balanced lattice rectangle in our example is then found by selecting $7(=\frac{s^m-1}{s-1})$ pairs of the form (line, point)(=(r-1)-flat, (c-1)-flat) such that each of the $7(=\frac{s^m-1}{s-1})$ points will occur an equal number of times on the 7 lines and an equal number of times on the 7 points. Since each line has in this case 3 points and 7, 3 and 1 have no greatest common divisor it follows immediately that each point will occur on 3 lines and will coincide once with 1 point. We can express the balanced lattice rectangle with the following matrix:

$$\begin{pmatrix} \gamma & \gamma & \delta & \delta \\ \alpha & \alpha & \alpha & \alpha \end{pmatrix} = \begin{pmatrix} \frac{s^r-1}{s-1} & \frac{s^r-1}{s-1} & \frac{s^c-1}{s-1} & \frac{s^c-1}{s-1} \\ \frac{s^m-1}{s-1} & \frac{s^m-1}{s-1} & \frac{s^m-1}{s-1} & \frac{s^m-1}{s-1} \end{pmatrix} = \begin{pmatrix} 3 & 3 & 1 & 1 \\ 7 & 7 & 7 & 7 \end{pmatrix}$$

This matrix is read as follows: There are 7 pairs such that each of the 7 effects is confounded in 3 arrangements or pairs with rows whereby there occur 3 effects in each row confounding and in 1 arrangement with columns such that 1 effect occurs in the column confounding. The existence of such balanced configurations has been proved by Raktoe (see Ph.D. thesis).

5. Construction of balanced lattices by means of cyclic collineations

In order to construct a balanced lattice rectangle for our 2^3 example, we need to specify 7 confounding schemes, which will yield the above matrix. If we try to do this in some logical manner (see Kempthorne's book) we land into some laborious counting and checking. To have an easier method Raktoe (see Ph.D. thesis) employed cyclic collineations.

A collineation defined on PG(m-1,s) is a point transformation represented by a m × m matrix Z with elements $z_{ij} \epsilon GF(s)$ such that $|Z| \neq 0$, with the understanding that Z and kZ represent the same collineation, where $k \epsilon GF(s)$ is not equal to zero. A collineation has the property that it takes points into points, lines into lines, flats into flats, and so on. Since $|A| \neq 0$ it follows that the columns (and rows) are independent, that is they generate a (m-1)-flat, i.e. PG(m-1,s) itself. Now if we can find a collineation Z such that $Z^{\left(\frac{s^m-1}{s-1}\right)} = I$, then we have a cyclic collineation of degree $\frac{s^m-1}{s-1}$; it then follows that from its respective $\frac{s^m-1}{s-1}$ powers we can construct a balanced lattice rectangle by simply reading off the first r columns to represent the row confounding and the remaining c=m-r columns to represent the column confounding. Raktoe has constructed cyclic collineations for all cases $s^m < 1000$ by means of an electronic computer.

For our $2^3 = 2^2 \cdot 2$ lattice rectangle such a collineation with its powers and the resulting confounding schemes is as follows:

Powers	Row confounding	Col. confounding		Treatment arrangement	
$Z = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$	C,A/AC	BC	$(C)_0,(A)_0$	0 0 0	0 1 0
			$(C)_1,(A)_0$	0 1 1	0 0 1
			$(C)_0,(A)_1$	1 0 0	1 1 0
			$(C)_1,(A)_1$	1 1 1	1 0 1
				$(BC)_0$	$(BC)_1$
$Z^2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$	BC,C/B	ABC	$(BC)_0,(C)_0$	0 0 0	1 0 0
			$(BC)_1,(C)_0$	1 1 0	0 1 0
			$(BC)_0,(C)_1$	0 1 1	1 1 1
			$(BC)_1,(C)_1$	1 0 1	0 0 1
				$(ABC)_0$	$(ABC)_1$
$Z^3 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$	ABC,BC/A	AB	$(ABC)_0,(BC)_0$	0 0 0	0 1 1
			$(ABC)_1,(BC)_0$	1 1 1	1 0 0
			$(ABC)_0,(BC)_1$	1 1 0	1 0 1
			$(ABC)_1,(BC)_1$	0 0 1	0 1 0
				$(AB)_0$	$(AB)_1$
$Z^4 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$	AB,ABC/c	AC	$(AB)_0,(ABC)_0$	0 0 0	1 1 0
			$(AB)_1,(ABC)_0$	1 0 1	0 1 1
			$(AB)_0,(ABC)_1$	1 1 1	0 0 1
			$(AB)_1,(ABC)_1$	0 1 0	1 0 0
				$(AC)_0$	$(AC)_1$

$$Z^5 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$ AC,AB/BC B

	$(B)_0$	$(B)_1$
$(AC)_0,(AB)_0$	0 0 0	1 1 1
$(AC)_1,(AB)_0$	0 0 1	0 1 1
$(AC)_0,(AB)_1$	1 0 1	0 1 0
$(AC)_1,(AB)_1$	1 0 0	0 1 1

$$Z^6 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$ B,AC/ABC A

	$(A)_0$	$(A)_1$
$(B)_0,(AC)_0$	0 0 0	1 0 1
$(B)_1,(AC)_0$	0 1 0	1 1 1
$(B)_0,(AC)_1$	0 0 1	1 0 0
$(B)_1,(AC)_1$	0 1 1	1 1 0

$$Z^7 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ A,B/AB C

	$(C)_0$	$(C)_1$
$(A)_0,(B)_0$	0 0 0	0 0 1
$(A)_1,(B)_0$	1 0 0	1 0 1
$(A)_0,(B)_1$	0 1 0	0 1 1
$(A)_1,(B)_1$	1 1 0	1 1 1

In the row confounding above the generated effect is written after the slash.

6. An ordered balanced set of lattice rectangles

Frequently the experimenter is unable to use a balanced lattice rectangle, since the number of arrangements are quite large for most cases. For example a $2^5 = 2^3 \cdot 2^2$ lattice rectangle requires 31 arrangements. In such cases it is desirable to order the balanced set in such a manner that it will enable the experimenter to choose the best q arrangements, where maximum ([m/r],[m/c]) $\leq q \leq \frac{s^m-1}{s-1}$; here [m/r] and [m/c] indicate the integers equal to or next larger than respectively m/2 and m/c. The usual "criterion" used for ordering is to select q arrangements such that each of the effects is confounded in as equal number of times with rows and as equal number of times in columns as possible.

An ordered scheme for our example is given by the vector of powers $(Z^7, Z, Z^2, Z^5, Z^3, Z^4, Z^6)$. The experimenter chooses the first two, first three, and so on, to get the best 2, best 3, and so on, arrangements.