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Optimal geometry in $\mathbb{R}^2$ for intercepting flow uniformly distributed over $[0,2\pi)$

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ABSTRACT

We show that in $\mathbb{R}^2$, if fluid flows linearly and in a direction uniformly distributed over $[0,2\pi)$, then a straight line will be the shape that averages the most flux per unit length. This result has implications for how *Brachynemurus* antlions distribute their chemical lures, as well as for how gillnets and mistnets should be deployed.

INTRODUCTION

Antlions exhibit two motile life stages: adults and larvae. Adults are winged and nocturnal, feeding on pollen and nectar. In contrast, larvae are stout and bristly, living in loose dry sand and soil. They feed exclusively on other arthropods, capturing them using pitfall traps and simple ambushes (Wheeler 1930, Powell and Hogue 1979).

As *Brachynemurus* antlion larvae travel through the sand, they generate trenches (Cain 1987). The trenches measure up to several meters long and can be from 1 to 5 mm wide – roughly the width of the larva. Recent results indicate that the trenches are laced with a chemical lure, released by the antlions and apparently derived from prey (Ladau, in preparation). The lure is attractive to *Pheidole* spp. ants, and it leads them to the larvae. Importantly, the lure is only effective once the ants have contacted the trenches (Ladau, in preparation).

One way that *Brachynemurus* could promote contact with their trenches is by shaping them efficiently. Consider a trench of length L and shape P. To an approaching ant, the trench's length will effectively equal the length of the projection of P onto a line perpendicular to the ant's direction of travel. If that length is denoted by a random variable X, then the most efficient trenches will be those that maximize $E[X \mid L]$ -- they will be the trenches that, on average, present the largest aspect to approaching ants.

Clearly, $E[X \mid L]$ will depend on the distribution of the ants' orientations. Cain (1987) concludes that *Brachynemurus* larvae are incapable of detecting both concentrations of prey and the directions of their movement. His conclusions are partially corroborated by Bahls and Deyrup (1988), who report that of the *Brachynemurus* species, only *B. nebulosus* congregate at ant nests. Excepting *B. nebulosus* then – which evidently lacks a chemical lure in addition to congregating at ant nests – *Brachynemurus* larvae appear naïve to the movements of prey, implying that to a larva, the ants' orientations are effectively distributed uniformly over 360°. Here we show under such an assumption, $E[X \mid L]$ is maximized when P is a straight line.

RESULTS AND DISCUSSION

Definitions. Let $t \in \mathbb{R}^1_+$, θ_0 be a random variable uniformly distributed over $[0,2\pi)$, and $r(\cdot)$ and $\varphi(\cdot)$ be functions such that $\mathbb{R}^1_+ \xrightarrow{r} \mathbb{R}^1$ and $\mathbb{R}^1_+ \xrightarrow{\varphi} \mathbb{R}^1$. Let P be a path defined by the polar parameterization $\langle r(t), \varphi(t) + \theta_0 \rangle$ over the interval $[0, t_f]$ with $\langle r(0), \varphi(0) \rangle = \langle 0,0 \rangle$.

Let $i \in \mathbb{N}$ and $n \in \mathbb{N}$. Consider inscribing a broken line S on P having $n+1$ segments, with the first and last segment having length $\Delta l/2$ and the remaining segments each having length Δl. Let s_i denote segment i and let X_i be the length of the projection of the contiguous halves of s_i and s_{i+1} on the x-axis. Let ϕ_i be the least non-negative residue modulo π of the angle made between s_i and s_{i+1}. Let X be the length of the projection of P onto the x-axis. These definitions are diagrammed in Figure 1.

Lemma 1. *If* P *is straight and has length* L, *then*

$$E[X] = \frac{2L}{\pi}. \tag{1}$$

Proof. Assume that P is straight, implying that

$$X = L|\cos\theta_0| \tag{2}$$

and

$$E[X] = E[L|\cos\theta_0|] \; . \tag{3}$$

Although θ_0 is distributed uniformly over $[0,2\pi)$, by symmetry it will suffice to treat θ_0 as if it is distributed uniformly over $[0,\pi)$. Hence, using

$$f_{\theta_0}(\theta_0) = \frac{1}{\pi},$$

we have that

$$E[X] = \int_0^{\frac{\pi}{2}} L\cos\theta_0 \cdot \frac{1}{\pi} d\theta_0 + \int_{\frac{\pi}{2}}^{\pi} -L\cos\theta_0 \cdot \frac{1}{\pi} d\theta_0 = \frac{2L}{\pi}.$$

Proposition 1. *If* $\phi_i = 0$ *or* $\phi_i = \pi$ *, then*

$$E[X_i] = \frac{\Delta l}{\pi}[1 + \sin(\frac{\phi_i}{2})]. \tag{4}$$

Proof. Consider the segment of S that corresponds with X_i. Since θ_0 is distributed uniformly over $[0,2\pi)$, the angle between s_{i+1} and the x-axis will also be distributed uniformly over $[0,2\pi)$. Let the latter angle be denoted by θ_i.

Now if $\phi_i = 0$, then X_i will correspond with a straight segment of S with length $L = \frac{\Delta l}{2}$. Since θ_i is distributed uniformly over $[0,2\pi)$, we can apply Lemma 1 to this segment taking $L = \frac{\Delta l}{2}$, obtaining

$$E[X_i] = \frac{\Delta l}{\pi}. \tag{5}$$

Since $\sin(0) = 0$, the desired result follows for $\phi_i = 0$. $\phi_i = \pi$ can be treated likewise, except that in this case the corresponding segment will have length Δl. Applying Lemma 1 yields

$$E[X_i] = \frac{2\Delta l}{\pi}. \tag{6}$$

The result is proved using the fact that $\sin(\frac{\pi}{2}) = 1$.

Proposition 2. *If* $0 < \phi_i < \pi$ *, then*

$$E[X_i] = \frac{\Delta l}{\pi}[1 + \sin(\frac{\phi_i}{2})]. \qquad (7)$$

Proof. Given $0 < \phi_i < \pi$, consider the segment of S corresponding with X_i. This segment will resemble a 'V,' with its orientation determined θ_i. The idea is to express X_i as a function of θ_i, and then take the expectation. In doing so, by symmetry it will suffice to examine only values of θ_i between 0 and π, while treating θ_i as if it were uniformly distributed over $[0, \pi)$.

Applying a case analysis to ϕ_i:

If $0 < \phi_i \leq \frac{\pi}{2}$, then

$$X_i = \begin{cases} \frac{\Delta l}{2}\cos\theta_i & \text{if } 0 \leq \theta_i \leq \frac{\pi}{2} - \phi_i \\ \frac{\Delta l}{2}\cos\theta_i + \frac{\Delta l}{2}\cos(\pi - \phi_i - \theta_i) & \text{if } \frac{\pi}{2} - \phi_i < \theta_i \leq \frac{\pi}{2} \\ \frac{\Delta l}{2}\cos(\pi - \phi_i - \theta_i) & \text{if } \frac{\pi}{2} < \theta_i \leq \pi - \frac{\phi_i}{2} \\ \frac{\Delta l}{2}\cos(\pi - \theta_i) & \text{if } \pi - \frac{\phi_i}{2} < \theta_i \leq \pi \end{cases}. \qquad (8)$$

If $\frac{\pi}{2} < \phi_i < \pi$, then

$$X_i = \begin{cases} \frac{\Delta l}{2}\cos\theta_i + \frac{\Delta l}{2}\cos(\pi - \phi_i - \theta_i) & \text{if } 0 \le \theta_i \le \frac{\pi}{2} \\ \frac{\Delta l}{2}\cos(\pi - \phi_i - \theta_i) & \text{if } \frac{\pi}{2} < \theta_i \le \pi - \frac{\phi_i}{2} \\ \frac{\Delta l}{2}\cos(\pi - \theta_i) & \text{if } \pi - \frac{\phi_i}{2} < \theta_i \le \frac{3\pi}{2} - \phi_i \\ \frac{\Delta l}{2}\cos(\pi - \theta_i) + \frac{\Delta l}{2}\cos(2\pi - \phi_i - \theta_i) & \text{if } \frac{3\pi}{2} - \phi_i < \theta_i \le \pi \end{cases} \quad (9)$$

Using,

$$f_{\theta_i}(\theta_i) = \frac{1}{\pi}$$

for $0 < \phi_i \le \frac{\pi}{2}$:

$$E[X_i] = \int_0^{\frac{\pi}{2}-\phi_i} \frac{\Delta l}{2}\cos\theta_i \cdot \frac{1}{\pi} d\theta_i + \int_{\frac{\pi}{2}-\phi_i}^{\frac{\pi}{2}} [\frac{\Delta l}{2}\cos\theta_i + \frac{\Delta l}{2}\cos(\pi - \phi_i - \theta_i)] \cdot \frac{1}{\pi} d\theta_i$$

$$+ \int_{\frac{\pi}{2}}^{\pi - \frac{\phi_i}{2}} \frac{\Delta l}{2}\cos(\pi - \phi_i - \theta_i) \cdot \frac{1}{\pi} d\theta_i + \int_{\pi - \frac{\phi_i}{2}}^{\pi} \frac{\Delta l}{2}\cos(\pi - \theta_i) \cdot \frac{1}{\pi} d\theta_i \quad (10)$$

$$= \frac{\Delta l}{\pi}[1 + \sin(\frac{\phi_i}{2})].$$

Similarly, for $\frac{\pi}{2} < \phi_i < \pi$:

$$E[X_i] = \int_0^{\frac{\pi}{2}} [\frac{\Delta l}{2}\cos\theta_i + \frac{\Delta l}{2}\cos(\pi - \phi_i - \theta_i)] \cdot \frac{1}{\pi} d\theta_i + \int_{\frac{\pi}{2}}^{\pi - \frac{\phi_i}{2}} \frac{\Delta l}{2}\cos(\pi - \phi_i - \theta_i) \cdot \frac{1}{\pi} d\theta_i$$

$$+\int_{\pi-\frac{\phi_i}{2}}^{\frac{3\pi}{2}-\phi_i} \frac{\Delta l}{2}\cos(\pi-\theta_i)\cdot\frac{1}{\pi}d\theta_i + \int_{\frac{3\pi}{2}-\phi_i}^{\pi} [\frac{\Delta l}{2}\cos(\pi-\theta_i)+\frac{\Delta l}{2}\cos(2\pi-\phi_i-\theta_i)]\cdot\frac{1}{\pi}d\theta_i$$

$$=\frac{\Delta l}{\pi}[1+\sin(\frac{\phi_i}{2})]. \tag{11}$$

Lemma 2. *For any* ϕ_i,

$$E[X_i]=\frac{\Delta l}{\pi}[1+\sin(\frac{\phi_i}{2})]. \tag{12}$$

Proof. Lemma 2 follows directly from Propositions 1 and 2.

Lemma 3.

$$E[X]\le \lim_{n\to\infty}\sum_{i=1}^{n}E[X_i]. \tag{13}$$

Proof. Clearly

$$X\le \lim_{n\to\infty}\sum_{i=1}^{n}X_i . \tag{14}$$

Taking expectations of both sides gives the desired result.

Proposition 3. *For* $A\equiv\{i:i\in\{1,2,3,....,n\},0\le\phi_i<\pi\}$, $i\in A$

$$E[X_i]<\frac{2\Delta l}{\pi}, \tag{15}$$

and for $B\equiv\{i:i\in\{1,2,3,....,n\},\phi_i=\pi\}$, $i\in B$

$$E[X_i]=\frac{2\Delta l}{\pi}. \tag{16}$$

Proof. Given sets A and B, it follows that

$$\sin(\frac{\phi_i}{2}) < 1, \forall i \in A \tag{17}$$

and that

$$\sin(\frac{\phi_i}{2}) = 1, \forall i \in B\,. \tag{18}$$

Adding one and multiplying by $\frac{\Delta l}{\pi}$ yields

$$\frac{\Delta l}{\pi}[1+\sin(\frac{\phi_i}{2})] < \frac{2\Delta l}{\pi}, \forall i \in A \tag{19}$$

and

$$\frac{\Delta l}{\pi}[1+\sin(\frac{\phi_i}{2})] = \frac{2\Delta l}{\pi}, \forall i \in B\,. \tag{20}$$

The proposition is proved by applying Lemma 2 to the left-hand sides of (*19*) and (*20*).

Proposition 4. *If* A *is non-empty,*

$$\sum_{i=1}^{n} E[X_i] < \frac{2n\Delta l}{\pi}. \tag{21}$$

Proof. Given A and B as defined above, assume A is non-empty. By definition, $\{A, B\}$ is a partition of $\{1,2,3,.....,n\}$. Therefore,

$$\sum_{i=1}^{n} E[X_i] = \sum_{i\in A} E[X_i] + \sum_{i\in B} E[X_i]\,. \tag{22}$$

By Proposition 3, we have that

$$\sum_{i \in A} E[X_i] < \sum_{i \in A} \frac{2\Delta l}{\pi} \tag{23}$$

and

$$\sum_{i \in B} E[X_i] = \sum_{i \in B} \frac{2\Delta l}{\pi}. \tag{24}$$

Hence, since A is non-empty, it follows that

$$\sum_{i=1}^{n} E[X_i] < \sum_{i \in A} \frac{2\Delta l}{\pi} + \sum_{i \in B} \frac{2\Delta l}{\pi} = \sum_{i=1}^{n} \frac{2\Delta l}{\pi} = \frac{2n\Delta l}{\pi}. \tag{25}$$

Theorem 1. *For any path* P *of length* L, *if* P *is straight then* E[X] *is maximized.*

Proof. Assume that P has length L, and that for the inscribed line S on P, set A is non-empty. By Proposition 4, it follows that

$$\lim_{n \to \infty} \sum_{i=1}^{n} E[X_i] < \lim_{n \to \infty} \frac{2n\Delta l}{\pi}. \tag{26}$$

Since by definition $\lim_{n \to \infty} n\Delta l = L$, we have that

$$\lim_{n \to \infty} \sum_{i=1}^{n} E[X_i] < \frac{2L}{\pi}. \tag{27}$$

Thus, if P is bent (i.e., A is non-empty), then by Lemma 3

$$E[X] < \frac{2L}{\pi}. \tag{28}$$

However, by Lemma 1, if P is straight, then $E[X] = \frac{2L}{\pi}$. Therefore, $E[X]$ is maximized when P is straight.

Since (1) $E[X \mid L]$ is maximized when P is straight and (2) ants' orientations are uniformly distributed over $[0,2\pi)$, *Brachynemurus* should be selected to deploy their lure in straight trenches. Consistent with the prediction, *Brachynemurus* larvae dig significantly straighter trenches than expected randomly (Ladau, in preparation). However, further research is needed to determine if straight trenches do indeed increase hunting success.

In addition to *Brachynemurus*, Theorem 1 has implications for how people deploy gillnets and mistnets. Although gillnets are used to capture fish while mistnets are used to capture birds, the two are structurally alike: a rectangle of netting, usually measuring at least 1 m tall and 3 m long. Both nets are suspended perpendicular to the seabed or ground, ensnaring victims that swim or fly by, respectively (Bub 1978, von Brandt 1991). As viewed from above, the nets can be deployed in essentially any shape; for instance, using posts or buoys, a semicircle, open triangle, or open square could be created. Theorem 1 implies that if the direction of fish or bird movement is unknown, then the optimal arrangement for the nets will be a straight line. Interestingly, such an arrangement already used by many fishers and birders (Bub 1978, von Brandt 1991).

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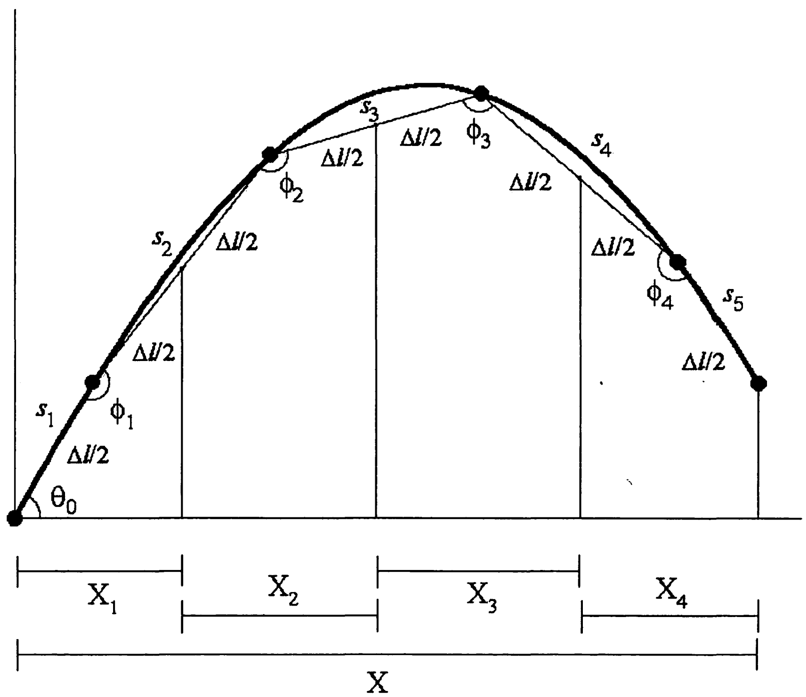
FIGURE LEGENDS

Figure 1. Diagram showing definitions used in proof. The case $n = 4$ is illustrated, with the path P given by the bold curved line.



s_1
ϕ_1
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