On the Maximum Number of Pairwise Orthogonal Row Frequency Squares *

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Abstract

Problems concerned with F-squares and complete sets of pairwise orthogonal F-squares have quite a history by now. In this paper concepts such as row frequency squares, column frequency squares, balancedness and orthogonality are introduced in connection with squares and vectors . Using these concepts and others subsequently introduced as well, the following results are obtained: (i) an upper bound for the maximum number of pairwise orthogonal row frequency squares; (ii) when s is a prime power a set of pairwise orthogonal row frequency squares of order s^k in s symbols can be constructed which achieves the bound in (i); (iii) the construction of the row frequency squares in (ii) is done by a simple algebraic-combinational process of embedding one suitable orthogonal array into another; (iv) a technique for producing row frequency squares of composite order is presented along with a lower bound for the number of pairwise orthogonal row frequency squares.

1 Introduction

Frequency squares are a generalization of Latin squares and were introduced by Finney (1945, 1946 (a,b)) and Freeman (1966). Subsequently, many researchers have contributed to the theory and construction of classes of F-squares which achieve certain properties, e.g. determining the best possible bound on the number of orthogonal F-squares with certain parameters and

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construction of the complete set for such squares. For a discussion on this topic as well as on orthogonal arrays see the book by Hedayat, Sloane and Stufken (1999). Contributions to resolving some of the problems in F-squares have, among others, been made by Hedayat and Seiden (1970), Hedayat, Raghavarao and Seiden (1975), Federer (1977, 2003), Mandeli, Lee and Federer (1981), Schwager, Federer and Raktoe(1984), Raktoe and Federer (1985), and Laywine(1993).

In Section 2, we introduce a selfcontained presentation of the basic concepts, definitions and all that is needed to understand the subsequent sections. Throughout, an algebraic-combinatorial approach is taken to make the reader understand the development in the paper without resorting to external structures, e.g. finite geometries, factorial designs and affine designs. Basic propositions are also established for subsequent use. In Section 3, we establish an upper bound for the number of pairwise orthogonal row frequency squares of order n in s symbols. It should be noticed that row frequency squares are a wider class which include frequency squares. In Section 4 it is shown that a class of pairwise orthogonal row frequency squares of order s^k in s symbols, where s is a prime or prime power, can be constructed which achieves the upper bound obtained in Section 3. This is done by the process of embedding an orthogonal array into a suitable orthogonal array of index unity. An example, illustrating the construction process is given. Further, the set with the maximum number of pairwise orthogonal row frequency squares of order s^k (s a prime or prime power) in s symbols constructed here includes a set with the maximum number of pairwise orthogonal frequency squares of order s^k in s symbols. This latter set of frequency squares was originally constructed by Hedayat, Raghavarao and Seiden (1975) using a different technique. In Section 5, if N(n, s) denotes the maximum number of pairwise orthogonal row frequency squares of order n in s symbols, then lower bounds for N(n, s) are obtained for any integers s $\geq$ 2 and n, with s dividing n. It is also established that N(n, s) $\geq$ 2 for all n, s with s $\geq$ 2. In Section 6 we present some thoughts about further research.

2 Preliminaries

Throughout the paper n and s will denote given positive integers with s dividing n. Boldface symbols such as **v** will denote column vectors and the transpose $\mathbf{v}'$ will refer to row vectors. In particular $\mathbf{1}_m$ will denote a m x 1 column vector each of whose entries is one.

Let S be any set of s elements. A n x n matrix L with entries from S will be called a SQ(n, s)-square. A SQ(n, s)-squares L will be called a *row frequency square (column frequency square)* or a RF(n, s)-square(CF(n, s)-square) iff each symbol in S occurs in each row (in each column) exactly n/s times. A *frequency square* or a F(n, s)-square is both a RF(n, s) and a CF(n, s)-square.

Hedayat, Raghavarao and Seiden (1975) studied some of the orthogonality aspects of F(n,s)-squares. The objective here is to study similar orthogonality aspects of the wider class of RF(n,s)-squares. Since the transpose of a RF(n, s)-square is a CF(n,s)-square, the results we obtain for RF(n, s)-squares are also valid for CF(n, s)-squares.

If m is some positive integer divisible by s then a m x 1 vector $\mathbf{v}$ with entries from S (over S) will be called *balanced* iff each element of S occurs in $\mathbf{v}$ exactly m/s times. We will be interested in n^2 x 1 vectors $\mathbf{v}$ over S presented as $\mathbf{v}' = (\mathbf{v}'_1, \mathbf{v}'_2, \ldots, \mathbf{v}'_n)$ where each $\mathbf{v}_i$ is a n x 1 vector; when $\mathbf{v}$ is presented in this component form we will call it a n-*component* vector.

A n-component vector $\mathbf{v}$ will be called:
1) *component balanced* iff each $\mathbf{v}_i$(i= 1, 2, ..., n) is balanced, 2) *entrywise balanced* iff for each j = 1, 2, ..., n the n x 1 vector formed by selecting the j-th entry from each $\mathbf{v}_i$ is balanced and 3) *totally balanced* iff it is both component and entrywise balanced.

Two n^2 x 1 vectors $\mathbf{u}$ and $\mathbf{v}$ over S (two SQ(n, s)-squares L_1, L_2) will be called *orthogonal*, in symbols $\mathbf{u} \perp \mathbf{v}$ ($L_1 \perp L_2$) iff when one is superimposed on the other, the ordered pairs of corresponding entries contains each of the s^2 possibilities exactly n^2/s^2 times. By extension a set $\{ \mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_w \}$ of n^2 x 1 vectors over S ($\{ L_1, L_2, \ldots, L_w \}$ of SQ(n, s)-squares) will be called *pairwise orthogonal* iff for all i, j with $i \neq j$, $\mathbf{c}_i \perp \mathbf{c}_j$ ($L_i \perp L_j$).

Let V(n, s) be the set of all n^2 x 1 n-component vectors over S and let M(n, s) be the set of all SQ(n, s)-squares. Then

a) for $\mathbf{v} \in$ V(n, s), say $\mathbf{v}' = (\mathbf{v}'_1, \mathbf{v}'_2, \ldots, \mathbf{v}'_n)$, define L($\mathbf{v}$) in M(n, s) to be the SQ(n, s)-square whose i-th row is $\mathbf{v}'_i$(i = 1, 2, ..., n),

b) for L$\in$ M(n, s), define $\mathbf{v}$(L) in V(n,s) by $\mathbf{v}'(L) = (\mathbf{r}'_1, \mathbf{r}'_2, \ldots, \mathbf{r}'_n)$ where $\mathbf{r}'_i$ is the i-th row of L(i = 1, 2, ... , n),

c) define the maps f : V(n, s) $\rightarrow$ M(n, s) and g : M(n,s) $\rightarrow$ V(n,s) by f($\mathbf{v}$)=L($\mathbf{v}$) and g(L) = $\mathbf{v}$(L).

The following Propositions reduce the study of orthogonality of the various kinds of squares in M(n, s) to the study of orthogonality of the corresponding kind of vectors in V(n,s).

Proposition 2.1.

(a) The maps f and g are one-to-one correspondences and inverses of each other,

(b) for $\mathbf{v} \in V(n, s)$, $L(\mathbf{v})$ is a RF(n, s)-square (respectively a CF(n, s)-square, F(n, s)-square) iff $\mathbf{v}$ is component balanced (respectively entrywise balanced, totally balanced), and

(c) for $L \in M(n, s)$, $\mathbf{v}(L)$ is component balanced (respectively entrywise balanced, totally balanced) iff L is a RF(n, s)-square (respectively a CF(n, s)-square, F(n, s)-square).

Proposition 2.2.

(a) Let $\mathbf{u}$ and $\mathbf{v}$ be n^2 x 1 vectors over S. Then $\mathbf{u} \perp \mathbf{v}$ implies that $\mathbf{u}$ and $\mathbf{v}$ are balanced,

(b) for $\mathbf{u}$ and $\mathbf{v}$ in V(n,s): $\mathbf{u} \perp \mathbf{v}$ iff $L(\mathbf{u}) \perp L(\mathbf{v})$, and

(c) for L_1, L_2 in M(n,s): $L_1 \perp L_2$ iff $\mathbf{v}(L_1) \perp \mathbf{v}(L_2)$.

Orthogonal arrays were introduced by Rao(1947). A N x k matrix B with entries from S is called an *orthogonal array* with s *levels*, *strength* t and index $\lambda (0 \leq t \leq k)$ if every N x t submatrix of B contains all t-tuples based on S exactly λ times as a row; such a matrix B will be referred to as an OA(N, k, s, t). Notice that $N=\lambda s^t$. Rao(1947) gave an implicit bound for k in the general case and for the case of strength t = 2 arrays, i.e. for an OA(N, k, s, 2) this bound is

$$k \leq \frac{(N-1)}{(s-1)}. \tag{2.1}$$

We emphasize that if $\mathbf{v}$ is a n^2 x 1 vector over S then $\mathbf{v}$ can be presented as a n-component vector $\mathbf{v}' = (\mathbf{v}'_1, \mathbf{v}'_2, ..., \mathbf{v}'_n)$ where the first n entries of $\mathbf{v}$ form $\mathbf{v}_1$, the next n entries of $\mathbf{v}$ form $\mathbf{v}_2$ and so on. In the next definitions the n^2 x 1 column vectors of the OA's considered are assumed to be n-component vectors in this way.

Let B be an OA(n^2, k, s, 2). Then B will be called a *row frequency orthogonal array* (*column frequency orthogonal array*), in symbols RFOA(n^2, k, s, 2) (CFOA(n^2, k, s, 2)) iff each column of B is component balanced (entrywise balanced). An OA(n^2, k, s, 2) will be called a *frequency orthogonal array* or an FOA(n^2, k, s, 2) iff it is both a row and a column frequency orthogonal array.

We will use the following notation: (i) POS(n, s, w) will denote a set of w pairwise orthogonal SQ(n,s)-squares, (ii) PORF(n, s, w) (POCF(n, s, w)) will denote a set of w pairwise orthogonal RF(n, s)-squares (CF(n, s)-squares) and POF(n, s, w) will denote a set of w pairwise orthogonal F(n, s)-squares.

From Propositions 2.1 and 2.2 we have the following: $\{L_1, L_2,..., L_w\}$ is a set of pairwise orthogonal SQ(n,s)-squares iff the set $\{\mathbf{v}(L_i)\colon 1 \leq i \leq w\}$ is a set of pairwise orthogonal n-component vectors, and this is so iff the matrix $B=[\mathbf{v}(L_1)\ \mathbf{v}(L_2)\ ...\ \mathbf{v}(L_w)\ ]$ is an OA(n^2, w, s, 2). Moreover, each L_i is a RF(n, s)-square iff each $\mathbf{v}(L_i)$ is a component balanced vector. Hence we have

Proposition 2.3.

(a) A set POS(n, s, w) and an OA(n^2, w, s, 2) coexist.

(b) A set PORF(n, s, w) and a RFOA(n^2, w, s, 2) coexist.

(c) A set POCF(n, s, w) and a CFOA(n^2, w, s, 2) coexist.

(d) A set POF(n, s, w) and a FOA(n^2, w, s, 2) coexist.

In view of inequality (2.1) and Proposition 2.3(a) we have

Corollary 2.1.

A necessary condition for the existence of a set POS(n, s, w) is

$$w \leq \frac{(n^2-1)}{(s-1)}. \tag{2.2}$$

In the next section an independent proof of inequality (2.2), which is a special case of inequality (2.1) will emerge.

3 Maximum Number of Orthogonal RF-squares

Hedayat et al (1975) showed that a necessary condition for the existence of a set POF(n, s, w) is

$$w \leq \frac{(n-1)^2}{(s-1)}. \tag{3.1}$$

The original proof of the inequality (3.1) was based on certain incidence matrices and their eigenvalues. Later, in their book Hedayat, Sloane and Stufken (1999) offer a somewhat different proof based on the dimension of a subspace U spanned by a set of real vectors of dimension n^2 x 1. This proof revolves around proving a key equation involving the dimension of U. They invoke the assumptions of orthogonality and of frequency squares in establishing this equation. A close examination of this proof shows that this key equation may be derived simply on the basis of orthogonality and does not require the frequency assumption at all. In the last step one needs to invoke the assumption of frequency squares to get the inequality (3.1).

In the lemma below we reproduce the argument of Hedayat et al (1999), adapted here in terms of OA's, to obtain the key equation involving the dimension of U based solely on the assumption of orthogonality. From this we will see that Corollary 2.1 follows as well as the main result of this section which is a bound on w for the existence of a set PORF(n, s, w).

The following notation, adapted from Hedayat et al (1999), will be used throughout the presentation below:

(a) Let L= $[\mathbf{c}_1\ \mathbf{c}_2\ \ldots\ \mathbf{c}_w]$ be any OA(n^2, w, s, 2).

(b) Corresponding to a column $\mathbf{c}_m$ of L, let the j-th entry, j = 1, 2, ...,n^2, of the n^2 x 1 vector $\mathbf{u}_{lm}$ be 1 if the j-th entry of $\mathbf{c}_m$ is the symbol l and -1/(s-1) otherwise, for l = 1, 2, ..., s and m = 1, 2, ..., w.

(c) Let U be the vector space spanned by the sw real vectors $\mathbf{u}_{lm}$, l = 1, 2, ..., s and m = 1, 2, ..., w. Let V be the orthogonal complement of U. We use dim(U), dim(V) to denote the dimensions U,V.

(d) Let U_m be the vector space spanned by the vectors $\mathbf{u}_{1m}$, $\mathbf{u}_{2m}$, ..., $\mathbf{u}_{sm}$, for m = 1, 2, ..., w.

Lemma 3.1.

The dimension of U, i.e. dim(U) = n^2-dim(V) = (s-1) w.

Proof. Since V is the orthogonal complement of U, it follows that dim(U) = n^2 - dim(V). Since L is an OA, the set $\{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_w\}$ is a pairwise orthogonal set of vectors and by Proposition 2.2(a) each $\mathbf{c}_m$ is balanced. From these two properties it may be verified that
(a) $\mathbf{u}'_{lm}\ \mathbf{u}_{l'm'} = 0$ for any l, l' = 1, 2, ...,s and m, m$'$ = 1, 2, ..., w with m $\neq$ m$'$, and (b) $\mathbf{u}'_{lm}\ \mathbf{u}_{l'm} = -n^2/(s-1)^2$, for any l, l' = 1, 2, ..., s with $l \neq l'$ and $\mathbf{u}'_{lm}\ \mathbf{u}_{lm} = n^2/(s-1)$. Thus $[\mathbf{u}_{1m}\ \ldots\ \mathbf{u}_{sm}]'\ [\mathbf{u}_{1m} \ldots\ \mathbf{u}_{sm}] = (n^2 s/(s-1)^2)[I_s - (1/s)J_s]$, where J_s is a s x s matrix all of whose entries is 1.

Since the rank of this matrix is s-1, we get dim(U_m) = s - 1. Hence dim(U) = $\sum_{m=1}^{w}$dim(U_m) = (s-1)w and the lemma is established.

Corollary 3.1

A necessary condition for the existence of an OA(n^2, w, s, 2) is $w \leq (n^2-1)/(s-1)$.

Proof. Since each column $\mathbf{c}_m$ is balanced it follows that $1'_{n^2}\mathbf{u}_{lm} = 0$ for all l = 1, 2, ..., s and m = 1, 2, ..., w. Hence $1_{n^2} \in$ V and dim V $\geq$ 1. Hence from Lemma 3.1, w(s-1) $\leq$ n^2 -1 establishing the corollary.

Theorem 3.1.

A necessary condition for the existence of a set PORF(n, s, w) is

$$w \leq (n^2 - n)/(s-1). \tag{3.2}$$

Proof. By Proposition 2.3(b), we may assume that a matrix L which is a RFOA(n^2, w, s, 2) exists. By Lemma 3.1, for this L, w(s-1) = n^2 - dim(V). Below we identify n independent n^2 x 1 vectors in V; this means that dim(V) $\geq$ n establishing the inequality (3.2). The desired n independent vectors are defined as follows: for each j = 1, 2, ..., n define an n-component n^2 x 1 vector $\mathbf{v}_j$ by

$$\mathbf{v}'_j = (\mathbf{v}'_{1j}, \mathbf{v}'_{2j}, ..., \mathbf{v}'_{nj}),$$

where for k = 1, 2, ..., n with k $\neq$ j the component $\mathbf{v}_{kj}$ is the zero vector and $\mathbf{v}_{jj} = \mathbf{1}_n$.

Clearly the set $\{\mathbf{v}_j$, $1 \leq j \leq n$ $\}$ is independent and since each column of L is component balanced it follows that $\mathbf{v}'_j \mathbf{u}_{lm} = 0$ for any j = 1, 2, ..., n, l = 1, 2, ..., s and m = 1, 2, ..., w.

The inequalities (2.2), (3.1) and (3.2) suggest the following definitions. Each of the sets (1) POS(n, s, w), (2) POF(n, s, w), and (3) PORF(n, s, w) will be called *complete* iff in case (1): w = (n^2-1)/(s-1), in case (2): w = $(n-1)^2$/(s-1) and in case (3): w = (n^2-n)/(s-1). Finally an OA(n^2, w, s, 2) is called *complete* iff w = (n^2-1)/(s-1); a RFOA(n^2, w, s, 2) will be called *complete* iff w = (n^2-n)/(s-1) and an FOA(n^2, w, s, 2) will be called *complete* iff w = $(n-1)^2$/(s-1).

4 Constructing a Complete Set of Orthogonal RF-squares

Hedayat et al(1975) constructed a complete set POF(s^m, s, $(s^m-1)^2$/(s-1)) when s is a prime or a prime power and m $\geq$ 1 is any integer. Their construction used ideas from symmetrical factorial design theory and projective geometry. In this section we construct a complete set PORF(s^m, s, (s^{2m}-s^m)/(s-1)) which includes a complete set POF(s^m, s, $(s^m-1)^2$/(s-1)) when s is a prime or a prime power. The key idea in the construction is that of embedding one suitable orthogonal array into another suitable orthogonal array to produce a row frequency orthogonal array.

Lemma 4.1.

Suppose that L_1 is an OA(n, k, s, 2) and L_2 is an OA(n^2, w, n, 2). Then there exists an n^2 x kw matrix L over S which is an OA(n^2, kw, s, 2). Moreover, (1) if L_2 is a RFOA(n^2, w, n, 2) then L is an RFOA(n^2, kw, s, 2), and (2) if L_2 is FOA(n^2, w, n, 2) then L is a FOA(n^2, kw, s, 2).

Proof. We give the construction for L. Let the set of symbols appearing in L_2 be from the set S = {1, 2, 3, ..., n}. Index the rows of the n x k matrix L_1 by the elements of S in the natural order. Let $\mathbf{c}_1$, $\mathbf{c}_2$, ..., $\mathbf{c}_w$ be the columns of L_2. Associate with each column $\mathbf{c}_j$(j = 1, 2, ..., w) a n^2 x k array A_j as follows: replace each entry in $\mathbf{c}_j$ by the row in L_1 indexed by that entry. In this way we obtain a n^2 x kw array L = [A_1 A_2 ... A_w]. Since L_2 is an array of index one and the columns of L_1 and L_2 are balanced the rest of the assertions about L are easily verified.

Corollary 4.1.

If there exists a set PORF(n, n, w) and an OA(n, k, s, 2) then there exists a set PORF(n, s, kw).

Remark 4.1.

It is well known that when s is a prime or a prime power and $k \geq 2$ a complete OA(s^k, (s^k-1)/(s-1), s, 2) can be constructed. Hedayat et al (1999) give three different constructions of this array. This array will be needed in the construction outlined below.

Proposition 4.1.

There exists a complete RFOA(s^{2k}, s^k, s^k, 2) which contains a complete FOA(s^{2k}, s^k-1, s^k, 2) as a subarray when s is a prime or a prime power and $k \geq 1$ is any integer.

Proof. By construction. Let m = s^k and let S = {0, 1, x, x^2, ..., x^{m-2}} be the elements of the Galois Field GF(m) of order m with primitive element x satisfying $x^{m-1} = 1$. For convenience we will write **1** below for the m^2 x 1 vector $\mathbf{1}_{m^2}$ all of whose entries is one. We define the m x 1 vector **c** by setting $\mathbf{c}'$ = (0 1 x x^2 ... x^{m-2}). For $b \neq 0$ in S, we define a m^2 x 1 vector $\mathbf{c}_{1b}$ as a m-component vector by setting $\mathbf{c}'_{1b}$ = (b$\mathbf{c}'$, $\mathbf{1}'$+b$\mathbf{c}'$, x$\mathbf{1}'$+b$\mathbf{c}'$, $x^2\mathbf{1}'$+b$\mathbf{c}'$, ..., $x^{m-2}\mathbf{1}'$+b$\mathbf{c}'$), where $x^j\mathbf{1}'$+b$\mathbf{c}'$ is the linear combination of the vectors $\mathbf{1}'$ and $\mathbf{c}'$ with the scalars x^j and b in S. We define one further m-component vector $\mathbf{c}_{01}$ of order m^2 x 1 by setting

$$\mathbf{c}'_{01} = (\mathbf{c}', \mathbf{c}', ..., \mathbf{c}').$$

Place $L_1 = [\mathbf{c}_{11}\ \mathbf{c}_{1x}\ \mathbf{c}_{1x^2}\ ...\ \mathbf{c}_{1x^{m-2}}\ ]$ and $L = [L_1\ \mathbf{c}_{01}]$. It may now be verified that L_1 is a FOA(s^{2k}, s^k-1, s^k, 2) and that L is a RFOA(s^{2k}, s^k, s^k, 2) and by construction both are complete.

Theorem 4.1.

There exists a complete set PORF(s^k, s , (s^{2k}-s^k)/(s-1)) which includes as a subset a complete set POF(s^k, s , (s^k-1)^2/(s-1)), when s is a prime or a power prime and $k \geq 1$ is any integer.

Proof. Case 1. k = 1. Then by Proposition 4.1 there exists a complete RFOA(s^2, s, s, 2) which includes a complete FOA(s^2, s-1, s, 2) as a subarray. Then by Proposition 2.3(b, d), Theorem 3.1 and inequality (3.1) this is equivalent to the existence of a complete set PORF(s, s, s) and a complete set POF(s, s, s-1).
Case 2. $k \geq 2$. By Remark 4.1 we know that there is a complete OA(s^k, (s^k-1)/(s-1), s, 2), say L_1. Let L_2 be the complete RFOA(s^{2k}, s^k, s^k ,2) constructed by the method described in Proposition 4.1 which includes the complete FOA(s^{2k}, s^k - 1, s^k, s, 2) as a subarray. Embed L_1 into L_2 as described in Lemma 4.1 to get the array L which by this lemma is a RFOA(s^{2k}, (s^{2k} - s^k)/(s-1), s, 2) and includes the FOA(s^{2k}, (s^k - 1)^2/(s-1), s, 2) as a subarray, and both of these arrays are clearly complete by construction. Then by Proposition 2.3(b,d), the theorem follows.

The following example illustrates the above construction method.

Example. Let k = s = 2. Let S = {0, 1, x, x^2} with $x^3 = 1$ be the elements of the Galois field GF(4). The complete arrays OA(4, 3, 2, 2) and RFOA(16, 4, 4, 2) are given below, the latter denoted by L_2 is constructed as described in Proposition 4.1.

	OA(4, 3, 2, 2)				$L_2 = [\mathbf{c}_{11}\ \mathbf{c}_{1x}\ \mathbf{c}_{1x^2}\ \mathbf{c}_{01}]$, where
	0	0	0	0	$\mathbf{c}'_{11} = (01xx^2, 10x^2x, xx^201, x^2x10)$
L_1:	1	0	1	1	$\mathbf{c}'_{1x} = (0xx^21, 1x^2x0, x01x^2, x^210x)$
	x	1	1	0	$\mathbf{c}'_{1x^2} = (0x^21x, 1x0x^2, x1x^20, x^20x1)$
	x^2	1	0	1	$\mathbf{c}'_{01} = (01xx^2, 01xx^2, 01xx^2, 01xx^2)$

We now embed L_1 into L_2 to obtain the array L which is a complete RFOA(16, 12, 2, 2) and the first nine columns of L is a complete FOA(16, 9, 2, 2). From L one can obtain the complete sets PORF(4, 2, 12) and POF(4, 2, 9).

The array RFOA(16, 12, 2, 2) is displayed below. The horizontal lines show each column of this array as a 16 x 1 4-component vector; the first nine of these columns are totally balanced and the final three are component balanced. The vertical lines show the three new columns formed from each column of L_2 upon the embedding of L_1 into L_2.

RFOA(16, 12, 2, 2)

	c_{11}			c_{1x}			c_{1x^2}			c_{01}	
0	0	0	0	0	0	0	0	0	0	0	0
0	1	1	1	1	0	1	0	1	0	1	1
1	1	0	1	0	1	0	1	1	1	1	0
1	0	1	0	1	1	1	1	0	1	0	1
0	1	1	0	1	1	0	1	1	0	0	0
0	0	0	1	0	1	1	1	0	0	1	1
1	0	1	1	1	0	0	0	0	1	1	0
1	1	0	0	0	0	1	0	1	1	0	1
1	1	0	1	1	0	1	1	0	0	0	0
1	0	1	0	0	0	0	1	1	0	1	1
0	0	0	0	1	1	1	0	1	1	1	0
0	1	1	1	0	1	0	0	0	1	0	1
1	0	1	1	0	1	1	0	1	0	0	0
1	1	0	0	1	1	0	0	0	0	1	1
0	1	1	0	0	0	1	1	0	1	1	0
0	0	0	1	1	0	0	1	1	1	0	1

5 On Pairwise Orthogonal RF-squares of Composite Order

Let N(n,s) denote the largest possible number of pairwise orthogonal RF(n, s)-squares. In Theorem 3.1 we showed that $N(n, s) \leq (n^2-n)/(s-1)$. Notice that when s = 1 this bound becomes infinite and that there is only one RF(n,1)-square which is self-orthogonal. Therefore, it will be convenient to put $N(n,1) = \infty$.

Let $X = (x_{ij})$ be a $RF(n_1, s_1)$-square and $Y=(y_{ij})$ a $RF(n_2, s_2)$-square in the symbols $S_1 = \{x_i{:}1 \leq i \leq s_1\}$ and $S_2 = \{y_j{:}1 \leq j \leq s_2\}$ respectively. For a symbol x_k in S_1 we define the *formal Kronecker product* $x_x \otimes Y$ to be the $SQ(n_2, s_2)$-square in the symbols $x_k \times S_2 = \{(x_k, y_j){:}\ 1 \leq j \leq s_2\}$ whose (i,j)-th entry is (x_k, y_{ij}), i,j = 1, 2, ..., n_2. By extension we define the *formal Kronecker product* $X \otimes Y$ as the $SQ(n_1n_2, s_1s_2)$-square in the symbols $S_1 \times S_2$, the cartesian product of S_1 with S_2, whose (i;j)-th entry is the SQ(n_2, s_2)-square $x_{ij} \otimes Y$, i,j = 1, 2, ..., n_1.

The following may be verified.

Proposition 5.1.

Let X and X_i(i= 1, 2) be RF(n_1, s_1)-squares, and Y and Y_i (i= 1, 2) be RF(n_2, s_2)-squares. Then

(a) $X \otimes Y$ is a RF(n_1n_2, s_1s_2)-square,

(b) if $X_1 \perp X_2$ and $Y_1 \perp Y_2$ then $X_1 \otimes Y_1 \perp X_2 \otimes Y_2$.

For squares of composite order we have the following

Proposition 5.2.

$N(n_1n_2, s_1s_2) \geq \min\{N(n_1, s_1), N(n_2, s_2)\}$ for $n_1 \geq 1$ and $n_2 \geq 1$.

Proof. Suppose that $N(n_i, s_i) = w_i$, i = 1, 2 and that $w_1 \leq w_2$. Then by Proposition 5.1, using the formal Kronecker product, we can construct at least w_1 pairwise orthogonal RF(n_1n_2, s_1s_2)-squares. This establishes the proposition.

Note that Proposition 5.2 extends to a set of any number of RF-squares of each order.

For a prime p and positive integers e and f such that e divides f we know from Theorem 4.1 that $N(p^f, p^e) = (p^{2f} - p^f)/(p^e - 1)$. However, in general we have the following

Corollary 5.1.

For a prime p and positive integers e, f, such that $e \leq f$, $N(p^f, p^e) \geq p^e$.

Proof. By Proposition 5.2, $N(p^f, p^e) \geq \min\{N(p^e, p^e), N(p^{f-e}, 1)\}$ Hence $N(p^f, p^e) \geq N(p^e, p^e) = p^e$, by Theorem 4.1 since p is a prime.

Corollary 5.2.

Let $n = p_1^{f_1} p_2^{f_2} \dots p_k^{f_k}$ be the factorization of n into distinct primes p_1, p_2, ..., p_k where $f_i \geq 1$ and let $s = p_1^{e_1} p_2^{e_2} \dots p_k^{e_k}$ with $0 \leq e_i \leq f_i$ for i = 1, 2, ..., k. Then

(1) $N(n,s) \geq \min\{p_i^{e_i}$: all i such that $e_i \neq 0\}$,

(2) $N(n,s) \geq 2$, for all $n \geq 2$ and $s \geq 2$,

(3) if for each i for which $e_i \neq 0$, e_i divides f_i, then $N(n,s) \geq \min\{(p_i^{2f_i} - p_i^{f_i})/(p_i^{e_i} - 1)$: all i such that $e_i \neq 0$ }

6 Discussion

As pointed out earlier, the results in this paper have relied on the algebraic-combinatorial approach with all aspects and definitions explained in a self-contained fashion so that the reader is unencumbered by external structures, such as finite geometries, affine designs and factorial designs. However, this does not mean that the results of the paper can not be achieved using these structures. It would not be surprising if subsequent researchers in this area can establish equivalent results by resorting to methods other than algebraic-combinatorial methods. Indeed, Federer(2003) has innovated the scene by coming up with methods not fitting all the conventional approaches above.

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