CONSTRUCTION OF F-SQUARES USING PERMUTATI0NS WITH PROPERTY A

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ABSTRACT

A method is presented for obtaining permutations of the integers -t through t such that differences between the permutations reproduce the integers -t through t. This is called property a. The method holds for prime numbers. Using these permutations, t F(n, 2) squares are constructed.

Keywords: Latin square; orthogonal; proportional orthogonality; semi-F-square.

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INTRODUCTION

An F-square is an n × n array with s symbols each of which appears n/s times in each row of the array and n/s times in each column of the array and is denoted as F(n, n/s). Such an F-square is called a regular F-square. When the s symbols occur equally frequent in a row (column) but not all symbols appear in a column (row), this called a semi-F-square (Federer, 2002). Non-regular F-squares have all symbols appearing in each row and in each column of the array but not equally frequent in a row and/or a column.

Anderson *et al.* (1974) presented the following theorem:

Theorem 2.1: Let n = 2t + 2. A set of r permutations of integers -t through t produces r - 1 orthogonal F(n, 2) squares if, when placed in an r × n - 1 array,

(a) *differences with the first row mod (n) reproduce -t through t, and*
(b) *differences of any other pair mod (t + 1) produce 0 one time and 1, 2, ..., t each two times.*

They present the following set of permutations for n = 6:

0:	-2	-1	0	1	2
1:	0	-2	1	-1	2
2:	-1	-2	2	1	0
3:	-2	0	2	-1	1

4:	2	-1	1	0	-2

Adding a matrix containing the letter k and with an extra column to the above set of permutations, 2t F-squares are constructed as follows:

Row:	k	k	k	k	k	k
Column:	k-2	k-1	k	k+1	k+2	k+3
Treatment 1:	k+0	k-2	k+1	k-1	k+2	k
Treatment 2:	k-1	k-2	k+2	k+1	k+0	k
Treatment 3:	k-2	k+0	k+2	k-1	k+1	k
Treatment 4:	k+2	k-1	k+1	k+0	k-2	k

Mod (6) is used for rows and columns and mod(3) for the treatments (symbols) in the F-square. This method constructs 2t = 4 combinatorially orthogonal F-squares. Anderson *et al.* (1974) then construct four more F-squares by interchanging the role of rows and columns to obtain a set of eight pair-wise mutually orthogonal F-squares, MOFS(6, 8). They show that this set cannot be extended. They also investigate several other methods of construction.

Anderson *et al.* (1974) presented the above permutations but gave no clue as to how such permutations are constructed. The purpose of this paper is to present a method for constructing permutations with property (a). The method holds for prime numbers.

METHOD OF CONSTRUCTING PERMUTATIONS

The algorithm for constructing t permutations with property (a) is:

Algorithm: Select a column from each one of the Latin squares in a complete set of orthogonal Latin squares of order p, p prime, to form a p $\times$ p-1 array. The differences between the elements of pairs of columns can be used to form 2t permutations of the integers -t through t. There are t distinct such permutations, the other t being duplicates. Several permutations are not of the form -t through t.

The first columns of MOLS(p, p - 1) orthogonal Latin squares for the primes 5, 7, and 11 and for the prime power 9 are:

p = 5	p = 7	p = 9	p = 11
1 1 1 1	1 1 1 1 1 1	1 1 1 1 1 1 1 1	1 1 1 1 1 1 1 1 1 1
2 3 4 5	2 3 4 5 6 7	2 9 8 7 6 5 4 3	2 3 4 5 6 7 8 9 10 11
3 5 2 4	3 5 7 2 4 6	3 2 9 8 7 6 5 4	3 5 7 9 11 2 4 6 8 10
4 2 5 3	4 7 3 6 2 5	4 3 2 9 8 7 6 5	4 7 10 2 5 8 11 3 6 9
5 4 3 2	5 2 6 3 7 4	5 4 3 2 9 8 7 6	5 9 2 6 10 3 7 11 4 8
	6 4 2 7 5 3	6 5 4 3 2 9 8 7	6 11 5 10 4 9 3 8 2 7
	7 6 5 4 3 2	7 6 5 4 3 2 9 8	7 2 8 3 9 4 10 5 11 6
		8 7 6 5 4 3 2 9	8 4 11 7 3 10 6 2 9 5
		9 8 7 6 5 4 3 2	9 6 3 11 8 5 2 10 7 4
			10 8 6 4 2 11 9 7 5 3

11 10 9 8 7 6 5 4 3 2

Taking differences between columns one and two and between columns one and three for p = 5 produce the 2 = t permutations given below. Differences between columns two and four and between three and four produce a duplicate of the permutation obtained for the permutation obtained from columns one and three. Differences between columns three and four reproduces the permutation obtained from one and two. The permutations obtained as differences between columns one and four and two and three do not contain the numbers -2 through 2. The following is used to construct F(6, 2) squares.

Row:	k	k	k	k	k	k
Column:	k-2	k-1	k	k+1	k+2	k+3
Treatment 1:	k+0	k-1	k-2	k+2	k+1	k
Treatment 2:	k+0	k-2	k+1	k-1	k+2	k

The two F(6, 2) squares obtained from the above are:

Column and treatment 1

0	1	2	3	4	5
1	2	1	0	0	2
0	2	0	2	1	1
2	1	0	1	0	2
0	0	2	1	2	1
2	1	1	0	2	0
1	0	2	2	1	0

Column and treatment 2

0	1	2	3	4	5
1	2	2	0	0	1
2	2	0	0	1	1
2	0	0	1	1	2
0	0	1	1	2	2
0	1	1	2	2	0
1	1	2	2	0	0

Each symbol occurs with itself eight times and with the other two symbols two times. This is called proportional orthogonality (Federer, 2002). The occurrences of the symbols in square 1 with those in square 2 are:

	Second square		
First square	0	1	2
0	8	2	2
1	2	8	2
2	2	2	8

Taking differences between columns one and two for n = 7, the permutation for treatment 1 is obtained. Differences between columns one and four results in the permutation for treatment 2. For treatment 3, we use differences between column two and column four. Differences between columns three and five, between columns three and six, and between columns five and six are duplicates of the previous set of three permutations. The following is used to construct the t = 3 F-squares:

Row:	k	k	k	k	k	k	k	k
Column:	k-3	k-2	k-1	k	k+1	k+2	k+3	k+4
Treatment 1:	k+0	k-1	k-2	k-3	k+3	k+2	k+1	k
Treatment 2:	k+0	k-3	k+1	k-2	k+2	k-1	k+3	k

Treatment 3: k+0 k-2 k+3 k+1 k-1 k-3 k+2 k

The t = 3 F(8, 2) squares constructed using the above are:

Column and treatment 1

0	1	2	3	4	5	6	7
1	3	2	1	0	0	3	2
3	2	0	3	2	1	1	0
1	0	3	1	0	3	2	2
3	2	1	0	2	1	0	3
0	0	3	2	1	3	2	1
2	1	1	0	3	2	0	3
0	3	2	2	1	0	3	1
2	1	0	3	3	2	1	0

Column and treatment 2

0	1	2	3	4	5	6	7
2	2	3	3	0	0	1	1
2	3	3	0	0	1	1	2
3	3	0	0	1	1	2	2
3	0	0	1	1	2	2	3
0	0	1	1	2	2	3	3
0	1	1	2	2	3	3	0
1	1	2	2	3	3	0	0
1	2	2	3	3	0	0	1

Column and treatment 3

0	1	2	3	4	5	6	7
1	3	1	2	0	0	2	3
0	2	0	2	3	1	1	3
0	1	3	1	3	0	2	2
3	1	2	0	2	0	1	3
0	0	2	3	1	3	1	2
3	1	1	3	0	2	0	2
3	0	2	2	0	1	3	1
2	0	1	3	3	1	2	0

Squares one and two are regular F-squares and square three is an irregular F-square. The occurrences of symbols in pairs of squares are:

	Second			
First	0	1	2	3
0	4	4	4	4
1	4	4	4	4
2	4	4	4	4
3	4	4	4	4

	Third			
First	0	1	2	3
0	8	4	0	4
1	4	8	4	0
2	0	4	8	4
3	4	0	4	8

	Third			
Second	0	1	2	3
0	4	4	4	4
1	4	4	4	4
2	4	4	4	4
3	4	4	4	4

Note that the first and third F-squares are not orthogonal.

For n = 11, differences between columns one and two, between one and six, between two and four, between three and seven, and between three and seven are used to obtain the t = 5 permutations for treatments 1 to 5 below. Differences between columns four and eight, between columns five and eight, between columns five and ten, between seven and nine, and between columns nine and ten are duplicates of the previous set of t = 5 permutations. The following is used to construct t = 5 F-squares:

Row:	k	k	k	k	k	k	k	k	k	k	k	k
Col:	k-5	k-4	k-3	k-2	k-1	k	k+1	k+2	k+3	k+4	k+5	k+6
Trt 1:	k+0	k-1	k-2	k-3	k-4	k-5	k+5	k+4	k+3	k+2	k+1	k
Trt 2:	k+0	k-5	k+1	k-4	k+2	k-3	k+3	k-2	k+1	k-1	k+5	k
Trt 3:	k+0	k-2	k-4	k+5	k+3	k+1	k-1	k-3	k-5	k+4	k+2	k
Trt 4:	k+0	k-3	k+5	k+2	k-1	k-4	k+4	k+1	k-2	k-5	k+3	k
Trt 5:	k+0	k-4	k+3	k-1	k-5	k+2	k-2	k+5	k+1	k-3	k+4	k

The t = 5 F(12, 2) squares obtained are:

Column and treatment 1

0	1	2	3	4	5	6	7	8	9	10	11
1	5	4	3	2	1	0	0	5	4	3	2

Column and treatment 2

0	1	2	3	4	5	6	7	8	9	10	11
3	3	4	4	5	5	0	0	1	1	2	2

Column and treatment 3

0	1	2	3	4	5	6	7	8	9	10	11
1	5	3	1	4	2	0	0	4	2	5	3

3	2	0	5	4	3	2	1	1	0	5	4
5	4	3	1	0	5	4	3	2	2	1	0
1	0	5	4	2	1	0	5	4	3	3	2
3	2	1	0	5	3	2	1	0	5	4	4
5	4	3	2	1	0	4	3	2	1	0	5
0	0	5	4	3	2	1	5	4	3	2	1
2	1	1	0	5	4	3	2	0	5	4	3
4	3	2	2	1	0	5	4	3	1	0	5
0	5	4	3	3	2	1	0	5	4	2	1
2	1	0	5	4	4	3	2	1	0	5	3
4	3	2	1	0	5	5	4	3	2	1	0

3	4	4	5	5	0	0	1	1	2	2	3
4	4	5	5	0	0	1	1	2	2	3	3
4	5	5	0	0	1	1	2	2	3	3	4
5	5	0	0	1	1	2	2	3	3	4	4
5	0	0	1	1	2	2	3	3	4	4	5
0	0	1	1	2	2	3	3	4	4	5	5
0	1	1	2	2	3	3	4	4	5	5	0
1	1	2	2	3	3	4	4	5	5	0	0
1	2	2	3	3	4	4	5	5	0	0	1
2	2	3	3	4	4	5	5	0	0	1	1
2	3	3	4	4	5	5	0	0	1	1	2

4	2	0	4	2	5	3	1	1	5	3	0
1	5	3	1	5	3	0	4	2	2	0	4
5	2	0	4	2	0	4	1	5	3	3	1
2	0	3	1	5	3	1	5	2	0	4	4
5	3	1	4	2	0	4	2	0	3	1	5
0	0	4	2	5	3	1	5	3	1	4	2
3	1	1	5	3	0	4	2	0	4	2	5
0	4	2	2	0	4	1	5	3	1	5	3
4	1	5	3	3	1	5	2	0	4	2	0
1	5	2	0	4	4	2	0	3	1	5	3
4	2	0	3	1	5	5	3	1	4	2	0

Column and treatment 4

0	1	2	3	4	5	6	7	8	9	10	11
2	4	1	4	1	3	0	0	3	5	2	5
0	3	5	2	5	2	4	1	1	4	0	3
4	1	4	0	3	0	3	5	2	2	5	1
2	5	2	5	1	4	1	4	0	3	3	0
1	3	0	3	0	2	5	2	5	1	4	4
5	2	4	1	4	1	3	0	3	0	2	5
0	0	3	5	2	5	2	4	1	4	1	3
4	1	1	4	0	3	0	3	5	2	5	2
3	5	2	2	5	1	4	1	4	0	3	0
1	4	0	3	3	0	2	5	2	5	1	4
5	2	5	1	4	4	1	3	0	3	0	2
3	0	3	0	2	5	5	2	4	1	4	1

Column and treatment 5

0	1	2	3	4	5	6	7	8	9	10	11
2	4	5	1	3	4	0	0	2	3	5	1
2	3	5	0	2	4	5	1	1	3	4	0
1	3	4	0	1	3	5	0	2	2	4	5
0	2	4	5	1	2	4	0	1	3	3	5
0	1	3	5	0	2	3	5	1	2	4	4
5	1	2	4	0	1	3	4	0	2	3	5
0	0	2	3	5	1	2	4	5	1	3	4
5	1	1	3	4	0	2	3	5	0	2	4
5	0	2	2	4	5	1	3	4	0	1	3
4	0	1	3	3	5	0	2	4	5	1	2
3	5	1	2	4	4	0	1	3	5	0	2
3	4	0	2	3	5	5	1	2	4	0	1

The third and fifth squares are irregular F-squares. The occurrences of symbols in pairs of squares are:

	Second					
First	0	1	2	3	4	5
0	8	2	4	4	4	2
1	2	8	2	4	4	4
2	4	2	8	2	4	4
3	4	4	2	8	2	4
4	4	4	4	2	8	2
5	2	4	4	4	2	8

	Third					
First	0	1	2	3	4	5
0	8	4	4	0	4	4
1	4	8	4	4	0	4
2	4	4	8	4	4	0
3	0	4	4	8	4	4
4	4	0	4	4	8	4
5	4	4	0	4	4	8

	Fourth					
First	0	1	2	3	4	5
0	4	6	2	4	2	6
1	6	4	6	2	4	2
2	2	6	4	6	2	4
3	4	2	6	4	6	2
4	2	4	2	6	4	6
5	6	2	4	2	6	4

	Fifth					
First	0	1	2	3	4	5
0	4	6	2	4	2	6
1	6	4	6	2	4	2
2	2	6	4	6	2	4
3	4	2	6	4	6	2
4	2	4	2	6	4	6

	Third					
2nd	0	1	2	3	4	5
0	4	4	2	8	2	4
1	4	4	4	2	8	2
2	2	4	4	4	2	8
3	8	2	4	4	4	2
4	2	8	2	4	4	4

	Fourth					
2nd	0	1	2	3	4	5
0	8	2	4	4	4	2
1	2	8	2	4	4	4
2	4	2	8	2	4	4
3	4	4	2	8	2	4
4	4	4	4	2	8	2

5	6	2	4	2	6	4

5	4	2	8	2	4	4

5	2	4	4	4	2	8

	Fifth					
Second	0	1	2	3	4	5
0	4	6	2	4	2	6
1	6	4	6	2	4	2
2	2	6	4	6	2	4
3	4	2	6	4	6	2
4	2	4	2	6	4	6
5	6	2	4	2	6	4

	Fourth					
Third	0	1	2	3	4	5
0	4	4	2	8	2	4
1	4	4	4	2	8	2
2	2	4	4	4	2	8
3	8	2	4	4	4	2
4	2	8	2	4	4	4
5	4	2	8	2	4	4

	Fifth					
Third	0	1	2	3	4	5
0	8	4	4	0	4	4
1	4	8	4	4	0	4
2	4	4	8	4	4	0
3	0	4	4	8	4	4
4	4	0	4	4	8	4
5	4	4	0	4	4	8

	Fifth					
Fourth	0	1	2	3	4	5
0	8	2	4	4	4	2
1	2	8	2	4	4	4
2	4	2	8	2	4	4
3	4	4	2	8	2	4
4	4	4	4	2	8	2
5	2	4	4	4	2	8

Except for the pairs one and three and three and five, the other eight pairs are proportionally orthogonal. The pairs of concurrences form symmetric matrices.

COMMENTS

From the above set of first columns of the MOLS(9, 8), it can be seen that permutations of -4 through 4 are not obtained by taking differences of columns. Taking the first column of the first Latin square, the second column of the second one, and the third column of the third Latin square, differences of columns did not produce permutations of -4 through 4 (Federer, 2002). Thus it is an open problem of how to construct permutations of -t through t for prime powers that satisfy property (a). Construction of permutations satisfying the conditions of the theorem is an open problem. Also, a method for other than prime numbers is needed for constructing orthogonal sets of F-squares.

LITERATURE CITED

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