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CONSTRUCTIONS OF ORTHOGONAL F(2k, q) SQUARES

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ABSTRACT

Anderson *et al.* (1974) present four methods of constructing pair-wise orthogonal F(2k, 2) squares, i.e. there are k symbols with each symbol appearing twice in a row and twice in the column of the 2k × 2k square. These methods are briefly discussed and illustrated. Comments are given about the generality of their results. A new method of constructing pair-wise orthogonal F(n, 2), n = 2k, squares is presented. In addition, one of the methods of the above cited authors involves the use of permutations of the numbers -t through t. They give no hint as to how such permutations are derived. A method for constructing these permutations is presented. To do this, use is made of a set of mutually orthogonal pair-wise Latin squares for n a prime number, MOLS(n, n - 1).

Key words: Orthogonal Latin squares; Permutation; Proportional orthogonality.

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INTRODUCTION

Anderson *et al.* (1974) present four methods for constructing a set of orthogonal F(2k, 2) squares where a F(2k, 2) square is a 2k × 2k square of k elements with each element occurring two times in each row and in each column. The four methods are:

(i) method of differences of permutations,
(ii) method of balanced incomplete block design,
(iii) method of difference composition, and
(iv) method of orthogonal arrays.

Each of the methods is briefly discussed and another method of constructing a set of orthogonal F- squares, Kronecker product method, is presented and illustrated for 2k from 6 to 50. In addition, a method of constructing permutations of the integers -t through t is given. This method involves the use of differences of the first columns of a mutually pair-wise orthogonal set of Latin squares, i.e., MOLS(n, n - 1) set, n a prime number. Anderson *et al.* (1974) make no mention of a method for constructing a set of permutations of -t through t which satisfy the conditions of their theorem 2.1. The F(n, 2) squares produced, are only proportionally orthogonal but are not orthogonal.

METHOD OF DIFFERENCES OF PERMUTATIONS

A set of 2t + 1 permutations from -t through t is selected in such a manner that differences of any pair of permutations reproduces 0 once and 1, 2, ... , t each two times modulo t. Anderson *et al.* (1974) present of five permutations for 2k = 6 that have this property (Federer, 2002). When their method was attempted for 2k = 10, only pairs of permutation were found which had this property. A set obtained by Federer (2002) is

-4	-3	-2	-1	0	1	2	3	4
-3	0	-4	1	4	-2	2	-1	3
-2	-4	1	3	0	-3	-1	4	2
-1	-2	2	-3	-4	0	4	3	1
0	-1	1	-4	-2	-3	3	2	4
-4	-2	-3	3	2	4	-1	1	0
-3	-4	1	3	0	-2	4	-1	2
-2	1	-3	-4	3	-1	2	4	0
-1	1	0	-4	-2	-3	2	4	3
0	-1	1	-3	-4	-2	2	4	3
-3	-1	1	-4	4	0	-2	3	2
-4	1	-1	2	-2	-3	4	0	3
0	-3	1	-4	2	-1	-2	4	3
-2	0	-1	-4	4	-3	2	1	3

Each of the last 13 permutations has the desired property with the first (ordered) permutation but no pair of these 13 has this property with each other. Federer (2002) raises the question as whether or not the method applies only to 2k = 6 as he was unable to find three permutations for 2k = 10 with the desired property.

METHOD OF BALANCED INCOMPLETE BLOCK DESIGN

This method uses an ordered F(2k, 2) square as a starting point. Then a resolvable balanced incomplete block design, BIBD, is obtained for v items in incomplete blocks of size s = 2 with b = v(v - 1)/2 incomplete blocks and r = bs/v. The 2k rows of the F(2k, 2) square are numbered from 1 to 2k. The rows of the r F-squares are obtained from the BIBD. To illustrate, let 2k = 6. Then the F(2k, 2) square and the BIBD for v = 6, s = 2, b = 15, r = 5 are

F(6, 2)	Rep1	Rep2	Rep3	Rep 4	Rep5
001122	1 2	1 3	1 4	1 5	1 6
001122	3 4	2 6	2 5	2 3	2 4
112200	5 6	4 5	3 6	4 6	3 5
112200					
220011					
220011					

The first columns of the five F-squares, which represent the row orderings, are:

Rep 1	Rep 2	Rep3	Rep 4	Rep 5
0	0	0	0	0
0	1	1	2	2
1	0	2	2	1
1	2	0	1	2
2	2	1	0	1
2	1	2	1	0

These orderings produce five orthogonal F(6, 2) squares.

This method did not appear to work to produce orthogonal F(10, 2) squares. For the BIBD used by Federer (2002), no orthogonal F-squares were produced but this may need further investigation.

METHOD OF DIFFERENCE COMPOSITION

For the difference composition method, a set of mutually orthogonal Latin squares of order p is required. Anderson *et al.* (1974) say to let p = 4t + 3, a prime number. They let p = 11 and were only able to produce a sequence of F(10, 2) squares where adjacent squares in the sequence were orthogonal but non-adjacent F-squares were not orthogonal. That is, they could not produce a triple of F(10, 2) squares which were orthogonal. They say that "in general, theorem 4.1 produces sets of sequences of F(p - 1, 2) squares such that adjacent pairs are orthogonal".

METHOD OF ORTHOGONAL ARRAYS

An orthogonal array, (N, k, n, t), of length N with k rows and n elements or symbols is required to use this method. Hence, the method does not appear usable to construct a set of unknown mutually orthogonal F-squares.

KRONECKER PRODUCT METHOD

Let J be a square matrix with all elements equal to one. Let F(2k, q) denote an F-squares with m = 2k/q symbols or elements. Also, let k be a prime number or power of a prime number. Let Li be one of the t Latin squares in the mutually orthogonal set MOLS(k, t), i = 1, 2, …, t. To illustrate for 2k = 6, the MOLS(3, 2) set is

L1 =	0	1	2	L2 =	0	1	2	J =	1	1
	1	2	0		2	0	1		1	1
	2	0	1		1	2	0			

The two F(6, 2) squares obtained are where * denotes Kronecker product:

J*L1 =	0	1	2	0	1	2	J*L2 =	0	1	2	0	1	2
	1	2	0	1	2	0		2	0	1	2	0	1
	2	0	1	2	0	1		1	2	0	1	2	0
	0	1	2	0	1	2		0	1	2	0	1	2
	1	2	0	1	2	0		2	0	1	2	0	1
	2	0	1	2	0	1		1	2	0	1	2	0

We now demonstrate the method for 2k = 8 through 50.

<u>2k = 8:</u>

The MOLS(4, 3) set of L1, L2, and L3 may be used to construct three orthogonal F(8, 2) as follows:

J*L1 J*L2 J*L3

<u>2k = 10:</u>

The MOLS(5, 4) set may be used to construct four F(10, 2) squares as follows;

J*L1 J*L2 J*L3 J*L4

2k = 12:

Since there are no orthogonal Latin squares of order six, no orthogonal F(12, 2) squares can be formed. However, using a J matrix of order three and an MOLS(4, 3) set, three mutually orthogonal F(12, 3) squares are formed as:

J*L1 J*L2 J*L3

Likewise, two orthogonal F(12, 4) squares may be constructed from an MOLS(3, 2) set.

2k = 14:

The MOLS(7, 6) set may be used to form six orthogonal F(14, 2) squares as follows:

J*L1 J*L2 J*L3 J*L4 J*L5 J*L6

2k = 16:

Seven orthogonal F(16, 2) squares may be formed using an MOLS(8, 7) set as:

J*L1 J*L2 J*L3 J*L4 J*L5 J*L6 J*L7

Also, three orthogonal F(16, 4) squares may be formed from the MOLS(4, 3) set as:

J*L1 J*L2 J*L3

2k = 18:

The MOLS(9, 8) set may be used to form eight orthogonal F(18, 2) squares as:

J*L1 J*L2 J*L3 J*L4 J*L5 J*L6 J*L7 J*L8

Also, two orthogonal F(18, 6) squares may be formed from the MOLS(3, 2) set with J of order six as:

J*L1 J*L2

2k = 20:

Two orthogonal F(20, 2) squares may be formed the MOLS(10, 2) set as

J*L1 J*L2

Three orthogonal F(20, 5) squares may be formed from the MOLS(4, 3) set as

J*L1 J*L2 J*L3

Four orthogonal F(20, 4) squares may be formed using the MOLS(5, 4) set as

J*L1 J*L2 J*L3 J*L4

<u>2k = 22:</u>

Use MOLS(11, 10) to obtain ten orthogonal F(22, 2) squares as

J*L1 J*L2 J*L3 J*L4 J*L5 J*L6 J*L7 J*L8 J*L9 J*L10

<u>2k = 24:</u>

Use the MOLS(12, 5) set to obtain five orthogonal F(24, 2) squares as:

J*L1 J*L2 J*L3 J*L4 J8L5

Use the MOLS(8, 7) set to obtain seven orthogonal F(24, 3) squares as:

J*L1 J*L2 J*L3 J*L4 J*L5 J*L6 J*L6 J*L7

Use the MOLS(4, 3) set to obtain three orthogonal F(24, 6) squares as:

J*L1 J*L2 J*L3

The MOLS(3, 2) set may be used to construct two orthogonal F(24, 8) squares as:

J*L1 J*L2

<u>2k = 26:</u>

Use the MOLS(13, 12) set to obtain 12 orthogonal F(26, 2) squares as;

J*L1 J*L2 J*L3 J*L4 J*L5 J*L6 J*L7 J*L8 J*L9 J*L19 J*L11 J*L12

<u>2k = 28:</u>

The MOLS(4, 3) set may be used to construct three orthogonal F(28, 7) squares. The MOLS(7, 6) set may be used to form six orthogonal F(28, 4) squares. The MOLS(14, t) set may be used to produce t orthogonal F(28, 2) squares.

<u>2k = 30:</u>

The MOLS(15, 4) set may be used to construct four orthogonal F(30, 2) squares. The MOLS(5, 4) set may be used to form four orthogonal F(30, 6) squares. The MOLS(10, 2) set may be used to construct two orthogonal F(30, 3) squares.

<u>2k = 32:</u>

Using the MOLS(16, 15) set, 15 orthogonal F(32, 2) squares may be formed. The MOLS(8, 7) set may be used to construct seven orthogonal F(32, 4) squares. The MOLS(4, 3) set may be used to construct three orthogonal F(32, 8) squares.

<u>2k = 34:</u>

Using the MOLS(17, 16) set, 16 orthogonal F(34, 2) squares may be constructed.

<u>2k = 36:</u>

The MOLS(9, 8) set may be used to construct eight orthogonal F(36, 4) squares. Using the MOLS(4, 3) set, three orthogonal F(36, 9) squares. The MOLS(3, 2) set may be used to construct two orthogonal F(36, 12) squares. Using the MOLS(18, t) set, t orthogonal F(36, 2) squares may be produced.

<u>2k = 38:</u>

The MOLS(19, 18) set may be used to produce 18 orthogonal F(38, 2) squares.

<u>2k = 40:</u>

Using the MOLS(4, 3) set, three orthogonal F(40, 10) squares may be formed. The MOLS(5, 4) set may be used to construct four orthogonal F(40, 8) squares. The MOLS(10, 2) set may be used to construct two orthogonal F(40, 4) squares. The MOLS(20, t) set may be used to produce t orthogonal F(40, 2) squares.

<u>2k = 42:</u>

For n = 3, 7, 14, or 21, the MOLS(n, t) set may be used to construct the corresponding orthogonal F(42, 42/n) squares.

<u>2k = 44:</u>

Using the MOLS(11, 10) set ten orthogonal F(44, 4) may be constructed. Likewise, the MOLS(4, 3) set may be used to construct three orthogonal F(44, 11) squares. The MOLS(22, t) set may be used to form t orthogonal F(44, 2) squares.

<u>2k = 46:</u>

The MOLS(23, 22) set may be used to construct 22 orthogonal F(46, 2) squares.

2k = 48:

The MOLS(4, 3), the MOLS(8, 7), the MOLS(16, 15) sets may be used to construct three orthogonal F(48, 12) squares, seven orthogonal F(48, 6) squares and 15 orthogonal F(48, 4) squares, respectively. The MOLS(24, t) set may be used to form t orthogonal F(48, 2) squares.

2k = 50:

The MOLS(25, 24) set may be used to construct 24 orthogonal F(50, 2) squares. The MOLS(5, 4) set may be used to construct four orthogonal F(50, 10) squares, and the MOLS(10, 2) set may be used to construct two orthogonal F(50, 5) squares.

ON A METHOD FOR CONSTRUCTING PERMUTATIONS WITH PROPERTY A

Property a of theorem 2.1 of Anderson *et al.* (1974) states that differences of permutations of -t through t (or 1 through 2t + 1) need to reproduce the numbers -t through t. It has been found that if one takes the first columns of a set of pair-wise mutually orthogonal Latin squares, MOLS(n, n - 1), the differences of the columns produces the numbers -t through t or a set similar to this. For n = 5, two sets of differences (see Appendix), i. e., A = -2 -1 0 1 2 and B = -3 -1 0 1 3 were obtained. The permutation A is the one referred to in theorem 2.1 of Anderson *et al.* (1974). The permutations obtained as differences between columns of the four squares are:

Square	2	3	4
1	A	A	B
2		B	A
3			A

Differences of columns 1 minus 2, 1 minus 3, 2 minus 4, and 3 minus 4 resulted in a permutation of -2, -1, 0, 1, and 2. Using these four permutations with the ordered permutation, four F(6, 2) squares were formed. These squares are not pair-wise orthogonal as the ratio of occurrences is 8 times for the same item and 2 times for the other items, i. e., they are proportionally orthogonal. The desired ratio is 4:4:4 rather than 8:2:2. Taking different columns for L1 to L4 did not change the ratio. It appears that this method obtains proportionally orthogonal F(6, 2) squares. The question arises as to the possibility of changing the construction method in such a manner as to obtain orthogonal F(6, 2) squares.

For n = 7, three types of permutations of numbers were obtained, i. e., A = -3 -2 -1 0 1 2 3, B = -4 -2 -1 0 1 2 4, and C = -5 -3 -1 0 1 3 5. The permutations obtained as differences between the columns of the six squares are:

Square

Square	2	3	4	5	6
1	A	B	A	B	C
2		B	A	C	B
3			C	A	B
4				B	A
5					A

For n = 11 and A = -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, the differences listed below were of the desired type of permutation, i.e. A:

	Square								
Square	2	3	4	5	6	7	8	9	10
1	A				A				
2			A						
3					A	A			
4							A		
5							A		A
6									
7								A	
8									
9									A

The four other symmetrical permutations obtained were B = -6 -4 -3 -2 -1 0 1 2 3 4 6, C = -7 -6 -3 -2 -1 0 1 2 3 6 7, D = -8 -5 -4 -2 -1 0 1 2 4 5 8, and E = -9 -7 -5 -3 -1 0 1 3 5 7 9.

APPENDIX

The first columns of an MOLS(n, n - 1) set are:

MOLS(5, 4)	MOLS(7, 6)	MOLS(9, 8)	MOLS(11, 10)
1 1 1 1	1 1 1 1 1 1	1 1 1 1 1 1 1 1	1 1 1 1 1 1 1 1 1 1
2 3 4 5	2 3 4 5 6 7	2 9 8 7 6 5 4 3	2 3 4 5 6 7 8 9 10 11
3 5 2 4	3 5 7 2 4 6	3 2 9 8 7 6 5 4	3 5 7 9 11 2 4 6 8 10
4 2 5 3	4 7 3 6 2 5	4 3 2 9 7 7 6 5	4 7 10 2 5 8 11 3 6 9
5 4 3 2	5 2 6 3 7 4	5 4 3 2 9 8 7 6	5 9 2 6 10 3 7 11 4 8
	6 4 2 7 5 3	6 5 4 3 2 9 8 7	6 11 5 10 4 9 3 8 2 7
	7 6 5 4 3 2	7 6 5 4 3 2 9 8	7 2 8 3 9 4 10 5 11 6
		8 7 6 5 4 3 2 9	8 4 11 7 3 10 6 2 9 5
		9 8 7 6 5 4 3 2	9 6 3 11 8 5 2 10 7 4
			10 8 6 4 2 11 9 7 5 3
			11 10 9 8 7 6 5 4 3 2

The MOLS(9, 8) set is the one given by Hedayat and Federer (1970).

To obtain the permutations, simply take differences as follows for n = 5:

1	2	3	4	5	1	2	3	4	5	1	2	3	4	5	1	3	5	2	4	1	3	5	2	4
1	3	5	2	4	1	4	2	5	3	1	5	4	3	2	1	4	2	5	3	1	5	4	3	2
0	-1	-2	2	1	0	-2	1	-1	2	0	-3	-1	1	3	0	-1	3	-3	1	0	-2	1	-1	2

1	4	2	5	3
1	5	4	3	2
0	-1	-2	2	1

Using these permutations, let us construct some F(6, 2) squares. Starting with the matrix

Row:	k	k	k	k	k	k
Column:	k - 2	k - 1	k + 0	k + 1	k + 2	k + 3
Treat 1:	k + 0	k - 1	k - 2	k + 2	k + 1	k
Treat 2:	k + 0	k - 2	k + 1	k - 1	k + 2	k
Treat 3:	k + 0	k - 2	k + 1	k - 1	k + 2	k
Treat 4:	k + 0	k - 1	k - 2	k + 2	k + 1	k

Letting k = 0, 1, 2, 3, 4, and 5, the first row designating the row, the second designating the column number, the third to sixth rows the treat i, i = 1, 2, 3, 4, the F(6, 2) squares, mod(3), are:

Column and treat 1

0	1	2	3	4	5
1	2	1	0	0	2
0	2	0	2	1	1
2	1	0	1	0	2
0	0	2	1	2	1
2	1	1	0	2	0
1	0	2	2	1	0

Column and treat 2

0	1	2	3	4	5
1	2	2	0	0	1
2	2	0	0	1	1
2	0	0	1	1	2
0	0	1	1	2	2
0	1	1	2	2	0
1	1	2	2	0	0

Column and treat 3

0	1	2	3	4	5
1	2	2	0	0	1
2	2	0	0	1	1
2	0	0	1	1	2
0	0	1	1	2	2
0	1	1	2	2	0
1	1	2	2	0	0

Column and treat 4

0	1	2	3	4	5
1	2	1	0	0	2
0	2	0	2	1	1
2	1	0	1	0	2
0	0	2	1	2	1
2	1	1	0	2	0
1	0	2	2	1	0

To obtain the permutations for n = 7, simply take differences of first columns of the MOLS(7, 6) set as:

1	2	3	4	5	6	7	1	2	3	4	5	6	7	1	2	3	4	5	6	7	1	2	3	4	5	6	7	1	2	3	4	5	6	7
1	3	5	7	2	4	6	1	4	7	3	6	5	2	1	5	2	6	3	7	4	1	6	4	2	7	5	3	1	7	6	5	4	3	2
0	-1	-2	-3	3	2	1	0	-2	-4	1	-1	4	2	0	-3	1	-2	2	-1	3	0	-4	-1	2	-2	1	4	0	-5	-3	-1	1	3	5

1 3 5 7 2 4 6   1 3 5 7 2 4 6   1 3 5 7 2 4 6   1 3 5 7 2 4 6   1 4 7 3 6 2 5
1 4 7 3 6 5 2   1 5 2 6 3 7 4   1 6 4 2 7 5 3   1 7 6 5 4 3 2   1 5 2 6 3 7 4
0 -1 -2 4 -4 2 1   0 -2 3 1 -1 -3 2   0 -3 1 5 -5 -1 3   0 -4 -1 2 -2 1 4   0 -1 5 -3 3 -5 -1

1 4 7 3 6 2 5   1 4 7 3 6 2 5   1 5 2 6 3 7 4   1 5 2 6 3 7 4   1 6 4 2 7 5 3
1 6 4 2 7 5 3   1 7 6 5 4 3 2   1 6 4 2 7 5 3   1 7 6 5 4 3 2   1 7 6 5 4 3 2
0 -2 3 1 -1 -3 2   0 -3 1 -2 2 -1 3   0 -1 -2 4 -4 2 1   0 -2 -4 1 -1 4 2   0 -1 -2 -3 3 2 1

The permutations from differences of first columns of the MOLS(9, 8) are not of the desired type. For example differences of the first three columns are:

1 2 3 4 5 6 7 8 9   1 2 3 4 5 6 7 8 9   1 9 2 3 4 5 6 7 8
1 9 2 3 4 5 6 7 8   1 8 9 2 3 4 5 6 7   1 8 9 2 3 4 5 6 7
0 -7 1 1 1 1 1 1 1   0 -6 -6 2 2 2 2 2 2   0 1 -7 1 1 1 1 1 1

Instead of taking first columns of L1 to L8, one could take different columns from each square. For the first three squares L1 to L3, this results in

1	2	3
2	3	6
3	6	8
4	4	4
5	9	7
6	8	5
7	1	2
8	5	9
9	7	1

The column differences are:

1 2 3 4 5 6 7 8 9   1 2 3 4 5 6 7 8 9   2 3 6 4 9 8 1 5 7
2 3 6 4 9 8 1 5 7   3 6 8 4 7 5 2 9 1   3 6 8 4 7 5 2 9 1
-1 -1 -3 0 -4 -2 6 3 2   -2 -4 -5 0 -2 1 5 -1 8   -1 -3 -2 0 2 3 -1 -4 6

These differences are not of the desired type of -4 through 4 but they are closer than using first columns differences. It appears that the method only works for n a prime number.

To obtain the permutations for n = 11, simply take differences of first columns of the Latin squares in the MOLS(11, 10) set as:

1 2 3 4 5 6 7 8 9 10 11   1 2 3 4 5 6 7 8 9 10 11   1 2 3 4 5 6 7 8 9 10 11
1 3 5 7 9 11 2 4 6 8 10   1 4 7 10 2 5 8 11 3 6 9   1 5 9 2 6 10 3 7 11 4 8
0 -1 -2 -3 -4 -5 5 4 3 2 1   0 -2 -4 -6 -3 1 -1 -1 6 4 2   0 -3 -6 2 -1 -4 4 1 -2 6 3

1 2 3 4 5 6 7 8 9 10 11	1 2 3 4 5 6 7 8 9 10 11	1 2 3 4 5 6 7 8 9 10 11
1 6 11 5 10 4 9 3 8 2 7	1 7 2 8 3 9 4 10 5 11 6	1 8 4 11 7 3 10 6 2 9 5
0 -4 -8 -1 -5 2 -2 -5 1 8 4	0 -5 1 -4 2 -3 3 -2 4 -1 5	0 -6 -1 -7 -2 3 -3 2 7 1 6
1 2 3 4 5 6 7 8 9 10 11	1 2 3 4 5 6 7 8 9 10 11	1 2 3 4 5 6 7 8 9 10 11
1 9 6 3 11 8 5 2 10 7 4	1 10 8 6 4 2 11 9 7 5 3	1 11 10 9 8 7 6 5 4 3 2
0 -7 -3 1 -6 -2 2 6 -1 3 7	0 -8 -5 -2 1 4 -4 -1 2 5 8	0 -9 -7 -5 -3 -1 1 3 5 7 9
1 3 5 7 9 11 2 4 6 8 10	1 3 5 7 9 11 2 4 6 8 10	1 3 5 7 9 11 2 4 6 8 10
1 4 7 10 2 5 8 11 3 6 9	1 5 9 2 6 10 3 7 11 4 8	1 6 11 5 10 4 9 3 8 2 7
0 -1 -2 -3 7 6 -6 -7 3 2 1	0 -2 -4 5 3 1 -1 -3 -5 4 2	0 -3 -6 2 -1 7 -7 1 -2 6 3
1 3 5 7 9 11 2 4 6 8 10	1 3 5 7 9 11 2 4 6 8 10	1 3 5 7 9 11 2 4 6 8 10
1 7 2 8 3 9 4 10 5 11 6	1 8 4 11 7 3 10 6 2 9 5	1 9 6 3 11 8 5 2 10 7 4
0 -4 3 -1 6 2 -2 -6 1 -3 4	0 -5 1 -4 2 8 -8 -2 4 -1 5	0 -6 -1 4 -2 3 -3 2 -4 1 6
1 3 5 7 9 11 2 4 6 8 10	1 3 5 7 9 11 2 4 6 8 10	1 4 7 10 2 5 8 11 3 6 9
1 10 8 6 4 2 11 9 7 5 3	1 11 10 9 8 7 6 5 4 3 2	1 5 9 2 6 10 3 7 11 4 8
0 -7 -3 1 5 9 -9 -5 -1 3 7	0 -8 -5 -2 1 4 -4 -1 2 5 8	0 -1 -2 8 -4 -5 5 4 -8 2 1
1 4 7 10 2 5 8 11 3 6 9	1 4 7 10 2 5 8 11 3 6 9	1 4 7 10 2 5 8 11 3 6 9
1 6 11 5 10 4 9 3 8 2 7	1 7 2 8 3 9 4 10 5 11 6	1 8 4 11 7 3 10 6 2 9 5
0 -2 -4 5 -8 1 -1 -8 -5 4 2	0 -3 5 2 -1 -4 4 1 -2 -5 3	0 -4 3 -1 -5 2 -2 5 1 -3 4
1 4 7 10 2 5 8 11 3 6 9	1 4 7 10 2 5 8 11 3 6 9	1 4 7 10 2 5 8 11 3 6 9
1 9 6 3 11 8 5 2 10 7 4	1 10 8 6 4 2 11 9 7 5 3	1 11 10 9 8 7 6 5 4 3 2
0 -5 1 7 -9 -3 3 9 -7 -1 5	0 -6 -1 4 -2 3 -3 2 -4 1 6	0 -7 -3 1 -6 -2 2 6 -1 3 7
1 5 9 2 6 10 3 7 11 4 8	1 5 9 2 6 10 3 7 11 4 8	1 5 9 2 6 10 3 7 11 4 8
1 6 11 5 10 4 9 3 8 2 7	1 7 2 8 3 9 4 10 5 11 6	1 8 4 11 7 3 10 6 2 9 5
0 -1 -2 -3 -4 6 -6 4 3 2 1	0 -2 7 -6 3 1 -1 -3 6 -7 2	0 -3 5 -9 -1 7 -7 1 9 -5 3
1 5 9 2 6 10 3 7 11 4 8	1 5 9 2 6 10 3 7 11 4 8	1 5 9 2 6 10 3 7 11 4 8
1 9 6 3 11 8 5 2 10 7 4	1 10 8 6 4 2 11 9 7 5 3	1 11 10 9 8 7 6 5 4 3 2
0 -4 3 -1 -5 2 -2 5 1 -3 4	0 -5 1 -4 2 8 -8 -2 4 -1 5	0 -6 -1 -7 -2 3 -3 2 7 1 6
1 6 11 5 10 4 9 3 8 2 7	1 6 11 5 10 4 9 3 8 2 7	1 6 11 5 10 4 9 3 8 2 7
1 7 2 8 3 9 4 10 5 11 6	1 8 4 11 7 3 10 6 2 9 5	1 9 6 3 11 8 5 2 10 7 4
0 -1 9 -3 7 -5 5 -7 3 -9 1	0 -2 7 -6 3 1 -1 -3 6 -7 2	0 -3 5 2 -1 -4 4 1 -2 -5 3
1 6 11 5 10 4 9 3 8 2 7	1 6 11 5 10 4 9 3 8 2 7	1 7 2 8 3 9 4 10 5 11 6
1 10 8 6 4 2 11 9 7 5 3	1 11 10 9 8 7 6 5 4 3 2	1 8 4 11 7 3 10 6 2 9 5
0 -4 3 -1 6 2 -2 -6 1 -3 4	0 -5 1 -4 2 -3 3 -2 4 -1 5	0 -1 -2 -3 -4 6 -6 4 3 2 1

1	7	2	8	3	9	4	10	5	11	6
1	9	6	3	11	8	5	2	10	7	4
0	-2	-4	5	-8	1	-1	8	-5	4	2

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1	10	8	6	4	2	11	9	7	5	3
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1	7	2	8	3	9	4	10	5	11	6
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1	10	8	6	4	2	11	9	7	5	3
1	11	10	9	8	7	6	5	4	3	2
0	-1	-2	-3	-4	-5	5	4	3	2	1

LITERATURE CITED

Anderson, D. A., W. T. Federer, and E. Seiden (1974). On the construction of orthogonal F(2k, 2) squares. BU-500-M in the Technical Report Series of the Department of Biological Statistics and Computational Biology, Cornell University, Ithaca, New York 14850.

Federer, W. T. (2002). On the construction of complete sets of F-squares of order n = 2k, n = 2^n k, and n = 4t with Latin squares. BU-1597-M in the Technical Report Series of the Department of Biological Statistics and Computational Biology, Cornell University, Ithaca, New York 14850.

Hedayat, A. and W. T. Federer (1970). On the equivalence of Mann's group automorphism method of constructing an O(n, n - 1) set and Raktoe's collineation method of constructing a balanced set of L-restrictional prime-power lattice designs. The Annals of Mathematical Statistics 41(5):1530-1540.