APPLICATION OF RAO'S GENERAL METHOD OF ANALYSIS TO A SIMPLE OR DOUBLE LATTICE DESIGN--W. T. Federer
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In 1947 C. R. Rao put forth a general method of analysis for incomplete block designs (Jour. Amer. Stat. Assoc. 42:541). The incomplete blocks designs are of the type that have v varieties in b incomplete blocks of k varieties with each variety repeated r times. Rao also considers the case where the number of replicates varies for the different varieties but this is not relevant to the simple lattice design for which $v = k^2$, $b = 2k$, and $r = 2$.

The general form of the analysis may be considerably simplified for the simple lattice design. Therefore, it was decided to set forth the analytical procedure and the specific parameters for the simple lattice design. An illustrative example of $k^2 = 9$ items or varieties in 2 replicates is presented also.

The simple lattice design represents two different groupings of the k^2 varieties such as the following:

Y grouping of varieties

X grouping of varieties	00	01	02	...	0, k-1
	10	11	12	...	1, k-1
	20	21	22	...	2, k-1
	⋮			...	...
	k-1, 0	k-1,1	k-1,2	...	k-1,k-1

Thus in the X grouping variety 00 appears with the k-1 varieties, 01, 02, ... 0,k-1 in an incomplete block while in the Y grouping 00 appears with varieties 10, 20, ..., k-1,0. It is obvious then that every one of the k^2 varieties will occur with 2(k-1) varieties in an incomplete block in the two groupings. Likewise, there will be $(k-1)^2$ varieties with which a given variety will not be associated in an incomplete block.

On the basis of "association" of varieties in the incomplete block, Rao sets up a system of associates. First associates (or logically zeroth associates) are varieties which do not appear together in an incomplete block. [Kempthorne (written correspondence 2/15/51) calls these second associates and vice versa.] Rao's second associates refer to pairs of varieties appearing together once in incomplete blocks.

The terminology zeroth, first, second, etc. associates will be adopted henceforth and these will be equivalent to Rao's first, second, third, etc. associates. The reason for this change is to have the degree of the associate refer to the number of times, λ_i, that varieties appear together in the incomplete blocks.

There are $n_0 = (k-1)^2$ zeroth associates and $n_1 = 2(k-1)$ first associates for any variety. $\lambda_0 = 0$ refers to a pair of varieties which are not compared in an incomplete block. $\lambda_1 = 1$ refers to a pair of varieties which appear together in one of the incomplete blocks. If a pair of varieties appeared together twice in the b incomplete blocks, then λ_2 would equal 2. However, the association of varieties in the incomplete blocks for the simple lattice design is either none or once and therefore there are only the two such parameters, $\lambda_0 = 0$ and $\lambda_1 = 1$.

One further type of parameter peculiar to the particular incomplete block design in question will be the number of associates in common among the various pairs of varieties with varying degrees of association. For a pair of varieties which are zeroth associates there are 4 parameters, $p^0_{00} = (k-2)^2$ = number of varieties in common among their zeroth associates, $p^0_{01} = p^0_{10} = 2(k-2)$ = number of varieties in common among their zeroth and first associates, and p^0_{11} = 2 = number of varieties in common among the first associates of the two varieties in question.

Likewise there are 4 parameters for a pair of varieties which are first associates. Thus, $p^1_{00} = (k-1)(k-2)$ = number of varieties in common between zeroth associates of the two varieties in question, $p^1_{01} = p^1_{10} = k-1$ = number of varieties in common between zeroth and first associates of the two varieties, and $p^1_{11} = k-2$ = number of varieties in common among their first associates.

The parameters for the simple lattice are summarized in Table 1. along with Rao's general notation and Kempthorne's notation.

Table 1. Parameters for a simple lattice design with the corresponding notation from Rao's paper and Kempthorne's letter.

Parameters for simple lattice		
Present terminology	Kempthorne's terminology	Rao's general terminology
k^2 = number of varieties	k^2	v
2 = " " replicates	2	r
2k = " " blocks	2k	b
k = " per block	k	k
$n_0 = (k-1)^2$	n_2	n_1
$\lambda_0 = 0$	λ_2	λ_1
$n_1 = 2(k-1)$	n_1	n_2
$\lambda_1 = 1$	λ_1	λ_2
$p^0_{ij} = \begin{pmatrix} (k-2)^2 & 2(k-2) \\ 2(k-2) & 2 \end{pmatrix}$	$p^2_{ij} = \begin{pmatrix} 2 & 2(k-2) \\ 2(k-2) & (k-2)^2 \end{pmatrix}$	p^1_{ij}
$p^1_{ij} = \begin{pmatrix} (k-1)(k-2) & (k-1) \\ (k-1) & (k-2) \end{pmatrix}$	$p^1_{ij} = \begin{pmatrix} (k-2) & (k-1) \\ (k-1) & (k-1)(k-2) \end{pmatrix}$	p^2_{ij}

With the above parameters for the simple lattice designs the following constants are computed:

$$\left.\begin{aligned} A_{01} &= 2k-1 \\ A_{11} &= p_{01}^{1} = k-1 \\ B_{01} &= 1 \\ B_{11} &= 2k-1 + (p_{00}^{0} - p_{00}^{1}) = k+1 \end{aligned}\right\} \quad \Delta = A_{01} B_{11} - A_{11} B_{01} = 2k^2$$

check

$$\left.\begin{aligned} A_{00} &= 2(k-1) \\ A_{10} &= -p_{01}^{0} = -2(k-2) \\ B_{00} &= -1 \\ B_{10} &= 2(k-1) - (p_{11}^{1} - p_{11}^{0}) \end{aligned}\right\} \quad \Delta = A_{00} B_{10} - A_{10} B_{00} = 2k^2$$

$$R = 2w + \frac{2w'}{k-1}$$

$$\Lambda_1 = w - w'$$

$$\Lambda_0 = 0$$

$$\left.\begin{aligned} A'_{01} &= (2k-1)\,w + w' \\ A'_{11} &= \Lambda_1\, p_{01}^{1} = (k-1)(w-w') \\ B'_{01} &= w-w' \\ B'_{11} &= A'_{01} + B'_{01}(p_{00}^{1} - p_{00}^{2}) \\ &= (k+1)w + (k-1)w' \end{aligned}\right\} \quad \Delta' = A'_{01} B'_{11} - A'_{11} B'_{01} = 2k^2 w(w+w')$$

check

$$\left.\begin{aligned} A'_{00} &= 2(k-1)w + 2w' \\ A'_{10} &= -(w-w')\, p_{01}^{0} = -2(k-2)(w-w') \\ B'_{00} &= -(w-w') \\ B'_{10} &= A'_{00} + B'_{00}\,(p_{11}^{1} - p_{11}^{0}) \\ &= (k+2)w + (k-2)\,w' \end{aligned}\right\} \quad \Delta' = A'_{00} B'_{10} - A'_{10} B'_{00} = 2k^2 w(w+w')$$

The variance of a mean difference for a pair of varieties which are zeroth associates is

$$\frac{2k\, B'_{10}}{\Delta'} = \frac{2}{k}\left\{\frac{1}{w+w'} + \frac{k-1}{2w}\right\}$$

and for a pair of varieties which are first associates is

$$\frac{2k\ B'_{11}}{\Delta'} = \frac{2}{k}\left\{\frac{2}{w+w'} + \frac{k-2}{2w}\right\}$$

The average standard error of a mean difference is

$$\frac{\frac{2k(k-1)^2}{2}\left(\frac{2}{k}\right)\left\{\frac{2}{w+w'} + \frac{k-2}{2w}\right\} + \frac{2k(2k-2)}{2}\left(\frac{2}{k}\right)\left\{\frac{1}{w+w'} + \frac{k-1}{2w}\right\}}{\frac{2k(k-1)^2}{2} + \frac{2k(2k-2)}{2}} = \frac{2}{k+1}\left\{\frac{2}{w+w'} + \frac{k-1}{2w}\right\}$$

since there are $k(k-1)^2$ comparisons among zeroth associates and $2k(k-1)$ comparisons among first associates.

The remainder of the analysis and adjustment of treatment means is easily comprehensible from an example. An artificial example of $k^2 = 9$ varieties in 2 replicates was constructed and is presented in Table 2. The variety totals and calculations for the analysis of variance are presented in Table 3 and the analysis of variance in Table 4. The adjusted means are presented in Table 5. The method of analysis follows the ordinary procedure for the analysis of a simple lattice design.

Table 2. Yields per plot for a simple lattice experiment with $k^2 = 3^2$ synthetic varieties in two replicates. Entry numbers in parenthesis.

Replicate I (Y arrangement)

(00)	(20)	(10)	(02)	(12)	(22)	(21)	(11)	(01)
8	5	3	3	2	6	3	7	3

Block Σ $(Y)_0 = 16$ $(Y)_2 = 11$ $(Y)_1 = 13$

Replicate Σ 40

Replicate II (X arrangement)

(21)	(20)	(22)	(10)	(11)	(12)	(01)	(02)	(00)
2	2	7	3	3	3	2	4	6

Block Σ $(X)_2 = 11$ $(X)_1 = 9$ $(X)_0 = 12$

Replicate Σ 32

Table 3. Total yields and other totals required for the analysis of the simple lattice experiment presented in Table 2.

	variety numbers and totals			$(A)_i$	$(X)_i$	$(A)_i-2(X)_i$	c'_x
	(00) 14	(01) 5	(02) 7	26	12	2	.24
	(10) 6	(11) 10	(12) 5	21	9	3	.37
	(20) 7	(21) 5	(22) 13	25	11	3	.37
Totals $(B)_j$	27	20	25	72	32	8	
$(Y)_j$	16	13	11	40			
$(B)_j-2(Y)_j$	-5	-6	3	-8		0	
c'_y	-.61	-.73	.37				

Table 4. Analysis of variance for the data of Table 2.

Randomized complete blocks analysis

Source of variation	d.f.	s.s.	m.s.
Replicates	1	3.56	3.56
Varieties or treatments	8	49.00	6.125
Residual	8	13.44	$1.680 = E'_e$
Total	17	66.00	

Simple lattice analysis

Source of variation	d.f.*	d.f.	s.s.	m.s.
Replicates	1	1	3.56	3.56
Varieties (ignor. blocks)	k^2-1	8	49.00	6.125
Blocks (elim. var.)	$2(k-1)$	4	8.22	$2.055 = E_b$
Among $(A)_i-2(X)_i$	$(k-1)$	2	1.111	
Among $(B)_j-2(Y)_j$	$(k-1)$	2	7.111	
Intrablock = residual	$(k-1)^2$	4	5.22	$1.305 = E_e$
Total	$2k^2-1$	17	66.00	

* General case

Table 5. Adjusted totals and means for the experiment in Table 2.

Variety number	Unadjusted totals	Sum of adjustments	Adjusted totals	Adjusted means
00	14	-.37	13.63	6.82
01	5	-.49	4.51	2.26
02	7	.61	7.61	3.80
10	6	-.24	5.76	2.88
11	10	-.36	9.64	4,82
12	5	.74	5.74	2.87
20	7	-.24	6.76	3.38
21	5	-.36	4.64	2.32
22	13	.74	13.74	6.87
Total	72	.03	72.03	36.02

For Rao's general method of analysis, the parameters for the example are:

$$r = 2,\ k^2 = 9,\ b = 6,\ k = 3$$

$$\lambda_0 = 0,\ n_0 = 4,\ \lambda_1 = 1,\ n_1 = 4$$

$$p^0_{ij} = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix} \qquad p^1_{ij} = \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix}$$

From Table 4, $w = \frac{1}{E_e} = 0.7663$ and $w' = \frac{1}{2E_b - E_e} = 0.3565$. The constants for the simple lattice are now computed as:

$$A_{01} = 5,\ A_{11} = 2,\ B_{01} = 1,\ B_{11} = 4, \text{ and } \Delta = 18$$

$$A_{00} = 4,\ A_{10} = -2,\ B_{00} = -1,\ B_{10} = 5, \text{ and } \Delta = 18$$

$$R = 2w+w' = 1.8891,\quad \Lambda_1 = w-w' = .4098,\quad \Lambda_0 = 0$$

$$A'_{01} = 4.1880,\ A'_{11} = .8196,\ B'_{01} = .4098,\ B'_{11} = 3.7782$$

$$A'_{00} = 3.7782,\ A'_{10} = -.8196,\ B'_{00} = -.4098,\ B'_{10} = 4.1880$$

$$\Delta = A'_{01}\,B'_{11} - A'_{11}\,B'_{01} = A'_{00}\,B'_{10} - A'_{10}\,B'_{00} = 2k^2w(w+w') = 15.487230.$$

Equipped with the A_{ij} and B_{ij}, it is now possible to begin construction of Table 6. The first column contains the variety number, the second column contains the variety totals α (see Table 5), and the third column contains the sum of the block totals containing a particular variety, for example variety 01 appears in blocks $(Y)_1$ and $(X)_0$ and the β value for variety 01 is 13+12=25.

The fourth column is computed from columns 2 and 3. For example the second γ value is

$$\gamma = k\alpha - \beta = 3(5) - 25 = -10.$$

The fifth column contains the $(k-1)^2$ varieties which are zeroth associates of variety α. The sixth column contains the sum of the γ values for variety α and its zeroth associates. The second ε value is computed as

$$-10 - 7 - 5 - 6 + 17 = -11$$

Table 6. Computations for simple lattice by Rao's method.

Var. no.	α	β	$\gamma=k\alpha-\beta$	δ	ε	η	A	B	C	D
00	14	28	14	11,12,21,22	25	2.5000	20.7102	60.5115	3.99921	6.8175
01	5	25	-10	10,12,20,22	-11	-2.1667	1.2495	33.9942	- .56162	2.2567
02	7	23	- 2	10,11,20,21	-16	0.3333	6.6669	30.8757	.98585	3.8042
10	6	25	- 7	01,02,21,22	-11	-1.3333	3.5484	33.9942	.06004	2.8783
11	10	22	8	00,02,20,22	31	.5000	13.9734	67.2483	1.99921	4.8175
12	5	20	- 5	00,01,20,21	-16	-.5000	3.2985	31.9452	.04665	2.8649
20	7	27	- 6	01,02,11,12	-15	-.8333	5.0277	30.2160	.56004	3.3783
21	5	24	- 9	00,02,10,12	- 9	-2.0000	1.6593	35.8833	-.50079	2.3175
22	13	22	17	00,01,10,11	22	3.5000	20.8701	60.3516	4.04668	6.8650
Total	72 =G	216 =kG	0	----	0	0.0000	77.0040 =kw'G	385.0200	10.63527	35.9999 =G/2

α = variety or treatment total

β = Sum of block totals in which variety α appears

$\gamma = k\alpha-\beta$

δ = zeroth associates of variety α

ε = Sum of the γ values for variety α and its zeroth associates

$$\eta = \frac{(B_{01} + B_{11})\gamma}{\Delta} - \frac{B_{01}\varepsilon}{\Delta} = \frac{(k+2)\gamma}{2k^2} - \frac{1}{2k^2}\varepsilon$$

$A = kw\alpha-(w-w')\beta$

B = Sum of A for variety α and its zeroth associates

$$C = \frac{(B'_{01} + B'_{11})A}{\Delta'} - \frac{B'_{01}\, B}{\Delta'} = \frac{[(k+2)w+(k-2)w']\, A - (w-w')B}{2k^2w(w+w')}$$

$$D = C + \bar{x} - \frac{\Sigma C}{k^2}$$

The η values in the seventh column are computed from the formula

$$\eta = \frac{(B_{01}+ B_{11})\,\gamma - B_{01}\,\varepsilon}{\Delta} = \frac{(k+2)\,\gamma - \varepsilon}{2k^2} .$$

For example, the first η value is

$$2.5000 = \frac{(3 + 2)(14) - (25)}{18} = \frac{45}{18}$$

With the above values, the analysis of variance may now be completed. The replicate sum of squares (Table 2) is

$$\frac{40^2 + 32^2}{9} - \frac{72^2}{18} = 3.56$$

The blocks within replicate (ignoring variety) sum of squares is

$$\frac{16^2 + 11^2 + 13^2 + 11^2 + 9^2 + 12^2}{3} - \frac{(40^2 + 32^2)}{9} = 5.78$$

The variety (eliminating block) sum of squares is

$$\frac{1}{k}\Sigma\Upsilon\eta = 51.44$$

and the variety (ignoring block) sum of squares is

$$\frac{\Sigma a_i^2}{2} - \frac{G^2}{2k^2} = 49.00$$

Therefore the blocks (eliminating variety) sum of squares is

$$5.78 - (49.00 - 51.44) = 8.22$$

as given in Table 4.

With the analysis of variance the values of $w = .7663$ and $w' = .3565$ are computed. The values of A'_{ij}, B'_{ij} and Δ' (see above) and the last 4 columns of Table 6 may now be calculated.

The first A value is computed from the formula $A = kw\alpha - (w-w')\beta = 3(.7663)14 - (.7663 - .3565)28 = 20.7102$. The B value, equal to the sum of the A values for a variety and its zeroth associates, for variety 00 is calculated as

$$20.7102 + 13.9734 + 3.2985 + 1.6593 + 20.8701 = 60.5115.$$

The C values are obtained from the formula

$$\frac{[(k+2)w + (k-2)w']A - (w-w')B}{2k^2w(w+w')}$$

$$= \frac{[5(.7663) + .3565]A - .4098B}{15.487230} = .27041634A - .02646051B,$$

the first C value being 3.99921.

The D values are the adjusted variety means and should correspond (within rounding errors) to those given in the last column of Table 5. The adjusted mean for variety 00 is

$$D = C + \left\{ \bar{x} - \frac{\Sigma C}{k^2} \right\} = 3.99921 + 2.81830 = 6.8175.$$

There are several partial checks available in the construction of Table 6. These are

$$\Sigma\alpha = G = \text{grand total}$$

$$\Sigma\beta = kG$$

$$\Sigma\gamma = \Sigma\varepsilon = \Sigma\gamma = \text{zero}$$

$$\Sigma A = kw'G$$

$$\Sigma B = [(k-1)^2 + 1]\Sigma A$$

$$\Sigma C = \frac{1}{2k^2 w(w+w')} \left\{ [(k+2)w + (k-2)w']\Sigma A - (w-w')\Sigma B \right\}$$

$$= w'G \left\{ \frac{w'(k-1)-w(k-3)}{2w(w+w')} \right\}$$

$$\Sigma D = G/2$$

Rao gives the following test of significance for an incomplete block design

$$\chi^2 = \frac{1}{k} \left\{ \Sigma AD - \frac{\Sigma A \Sigma D}{k^2} \right\} = 37.4808$$

with $k^2-1 = 8$ degrees of freedom. From a preliminary examination, it does not appear that the above χ^2 value is algebraically or numerically equal to the sum of squares of adjusted totals divided by the average effective error variance

$$\frac{2(\Sigma D^2 - (\Sigma D)^2/k^2)}{\frac{2}{k+1}\left\{ \frac{2}{w+w'} + \frac{k-1}{2w} \right\}} = \frac{2(25.65545)}{1.5431165} = 33.2502$$

as is the case for a balanced lattice design.