ESTIMATORS FOR COMPONENTS OF VARIANCE

BU-144-M W. T. Federer April, 1962

ABSTRACT

Four different estimators are available in statistical literature for computing the pair of variance components in a one-way classification. None of these four are admissible estimators. Two new sets of estimators for the pair of variance components are presented. These estimators result in non-negative estimates of the variance components and are analytic functions, and hence are admissible on these counts, whereas the presently available estimators are not. The method extends to more complex classifications.

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In a one-way classification of the following form:

Source of variation	d.f.	m.s.	Average mean square
Among groups	g-1	B	$\sigma_\epsilon^2+n\sigma_\beta^2$
Within groups	N-g	W	σ_ϵ^2
Total corrected for mean	N-1	T	$\sigma_\epsilon^2 + m\sigma_\beta^2$

(where N = total number of observations and g = total number of groups)

the most commonly used estimators for σ_ϵ^2 and σ_β^2 are:

$$\hat{\sigma}_\epsilon^2 = W$$

and

$$\hat{\sigma}_\beta^2 = \frac{B-W}{n} \qquad (1)$$

The range of estimates are:

$$0 \le \hat{\sigma}_\epsilon^2 < \infty$$

and

$$-\infty < \hat{\sigma}_\beta^2 < \infty \quad ,$$

where

$$0 \le W < \infty$$

and

$$0 \le B < \infty$$

Since a negative estimate of σ_β^2 is undefined the above estimator for σ_β^2 is inadmissible.

A second set of estimators that has been used is the following:

$$\left.\begin{aligned} \hat{\sigma}^2_\epsilon &= W \\ \text{and} \quad \hat{\sigma}^2_\beta &= \frac{B-W}{n} \quad \text{for} \quad B \geq W \\ &= 0 \quad \text{for} \quad B < W \end{aligned}\right\} \tag{2}$$

It has been shown that the estimator of σ^2_β is inadmissible because it is not an analytic function (i.e., not differential in a region near zero).

A third set of estimators has been proposed by Herbach [Annals of Math. Stat. 30:939, 1959]. He used the maximum likelihood principle to obtain variance component estimators for balanced one-way classifications (i.e., N-g=g(r-1)) of the form:

$$\left.\begin{aligned} \tilde{\sigma}^2_\epsilon &= W \quad \text{and} \quad r\tilde{\sigma}^2_\beta = (1-1/g)B-W \;, \quad \text{for} \;\; (1-1/g)B \geq W \\ \tilde{\sigma}^2_\epsilon &= T/rg \quad \text{and} \quad \tilde{\sigma}^2_\beta = 0 \;, \quad \text{for} \;\; (1-1/g)B < W \end{aligned}\right\} \tag{3}$$

Thompson [Annals of Math. Stat. 33:273, 1962] has used a restricted maximum likelihood estimator of the form:

$$\left.\begin{aligned} \approx{\sigma}^2_\epsilon &= W \quad \text{and} \quad \approx{\sigma}^2_\beta = (B-W)/r \quad \text{for} \quad B \geq W \\ \approx{\sigma}^2_\epsilon &= T/(rg-1) \quad \text{and} \quad \approx{\sigma}^2_\beta = 0 \quad \text{for} \quad B < W \end{aligned}\right\} \tag{4}$$

It appears that (3) and (4) will have the same difficulties regarding admissibility as (2). Therefore, two additional estimators for σ^2_β are listed below which are not deficient in the manner that estimators (1) to (4) are with regards to admissibility. It will be necessary, however, to verify whether they are or are not admissible estimators on other counts. For the first estimator for σ^2_β consider the following:

$$\overset{+}{\sigma}^2_\beta = \frac{(B-W)}{n} + \frac{W}{n}e^{-\alpha B} = \frac{W}{n}(F-1+e^{-\alpha B}) \quad , \tag{5}$$

where $F=B/W$, α is a constant, and $0 < \alpha \leq W$. The second estimator for σ_β^2 is:

$$\bar{\sigma}_\beta^2 = \frac{W}{n}(F-1+e^{-\delta F}) \quad , \tag{6}$$

where δ/W in (6) = α in (5), F is defined as for (5), and $0 < \delta \leq 1$.

The method extends directly to more complex classifications.