Can we get a head start on head lice?

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Abstract

Head lice, *Pediculus humanus capitis*, a common parasite found world-wide, affect children most often. School staffs frequently lack the necessary knowledge and expertise to control lice epidemics. The observed properties of the distribution of lice in humans result from a "macroparasitic" model. In this article, we provide a simple model to better understand the mechanisms behind the distribution of head lice among humans. We also provide an epidemiological model that models the transmission dynamics of lice in humans. The basic reproductive number is computed and involves two terms: the transmission of lice to of susceptible individuals by those with few and many lice, respectively. We establish conditions for the eradication and persistence of lice in humans.

1 Introduction

Pediculus humanus capitis, more commonly referred to as head lice, are cosmopolitan. Although they can infest anybody, it appears that some human groups are more susceptible to infestation than others. For example, children are infested more than adults and females more than males; individuals with clean heads are more likely to be infested than those with dirty heads or heads with dandruff; and large families are more likely to suffer from lice than small

ones, perhaps due to the crowding associated with large families rather than to socio-economic status (Chunge 1991, 96).

Lice are ectoparasitic insects because they live–that is, they breed and feed–on their host. Rather than being infected by lice, hosts are infested with lice. Lice are wingless, and thus, are unable to fly. Lice also lack the ability to jump. Their means of movement is through crawling. Human lice belong to the order Anoplura, which consists of about 560 blood-sucking species that live solely on mammals (Chunge 1991, 197). Although there are three types of lice –head lice (*Pediculus humanus capitis*), body lice (*Pediculus humanus humanus*), and crab lice (*Phthirus pubis*)– in this report we concentrate on head lice, which we will simply refer to as "lice."

The life history of a louse is divided into three stages: first, the egg-stage, during which a louse goes through an incubation period; the larval stages; and finally, the mature adult stage. Eggs hatch after an average incubation period of 7 days at 32-35 degrees Celsius (Kluge 1980, 9). Larvae reach maturity in an average of 8-9 days (Kluge 1980, 10). The life span of adult lice is approximately 30 days (Kluge 1980, 11).

Lice are transmitted primarily through direct physical contact with an infested individual. Although they are not primarily responsible for the spread of any disease, they are a cause for considerable social concern (Chunge 1991, 196). Children are affected most often, perhaps due to the high rate of physical interaction that occurs in schools. Students are especially susceptible to infestation. The following chart presents data on lice infestation in elementary schools in Poland during 1990-91. The data illustrate the potential severity of lice infestation (Wegner 1994, 221).

Lice infestation in children

Type of school	Total number of children	Number of infested children	Percent of infested children
Ordinary	23260	706	3.0
Special	644	44	6.8
Village	2580	120	4.2
Nursery	1316	11	0.8

Table 1: Lice infestation in children from elementary schools in Poland. Modified from Wegner, Racewicz, and Stanczak 1994.

Lice are readily transmitted if infested individuals are left untreated, and therefore, an

epidemic can easily erupt. Even though many people are insensitive to the bite of a louse and, in fact, may neither feel it nor show any reaction to it, infestation remains a community concern. Infested individuals are often unaware of their condition, especially during the first few weeks of infestation, when lice are fewer in number. For individuals infested with few lice, the symptoms are usually mild. However, overexposure to lice saliva as a result of hundreds of bites may result in itchy, allergic reactions. Sometimes, severe scratching leads to secondary bacterial or fungal infection. Thus, in more sensitive individuals or in cases of advanced infestation, louse bites result in itching which may result in swelling and loss of sleep. A child's education may be negatively affected as a consequence (Buxton 1990, 54).

In this report, we focus on the question of when a population should be treated for lice in order to optimize the effects of treatment. We use an epidemiological model in which the population is subdivided into three different groups: susceptible; infested with some lice; and infested with a large number of lice. A model of lice population growth in their natural habitat was presented in 1991 by B.V. Boev, V.K. Barabash, and I.V. Tarasevich. A model that relates the number of lice to individuals and attempts to explain the advantages of early prevention, however, has not yet been proposed. We propose such a model in this article.

Our study includes two main trajectories of investigation. first, we introduce an epidemiological model that examines lice-host dynamics as an "infectious" disease. We assume that lice are transmitted through contacts between lice-infested individuals and those without lice. Second, we look at a rough characterization of the temporal distribution of lice in a population. We develop a model that keeps track of the number of individuals stratified by the number of adult lice, and we observe changes in the mean-lice load and the variance of the population of lice in humans. The rest of this paper is organized as follows: Section 2 introduces a model for the transmission dynamics of lice in humans. An analysis of the model's stability follows. Also a special case study dealing with an infested Nigerian population is studied. Section 3 introduces a macroparasitic model in order to study the mechanism behind the observed distribution of lice in humans. Section 4 summarizes some results of the macroparasitic model discussed in the previous section. Section 5 lists our conclusions and suggests directions for future work.

2 An epidemiological model for the transmission dynamics of lice

Our epidemiological model consists of three differential equations. We let S, P_1, and P_2 represent the population size of susceptible individuals; infested with few adult lice; and individuals infested with a large number of adult lice, respectively. The transmission dynamics of lice in humans can be modeled by the following nonlinear system of ordinary differential equations:

$$\frac{dS}{dt} = \mu N - \beta_1 S\frac{P_1}{N} - \beta_2 S\frac{P_2}{N} - \mu S + \gamma_1 P_1 + \gamma_2 P_2 \quad (1)$$

$$\frac{dP_1}{dt} = \beta_1 S\frac{P_1}{N} + \beta_2 S\frac{P_2}{N} - (\mu + \gamma_1)P_1 - \omega P_1 \quad (2)$$

$$\frac{dP_2}{dt} = \omega P_1 - (\mu + \gamma_2)P_2 \quad (3)$$

where N, the total population is given by

$$N = S + P_1 + P_2. \quad (4)$$

Susceptible individuals can become infested in two ways: via contacts with individuals with few lice (P_1) or via contacts with individuals with "many lice" (P_2). The probability of becoming infested by an individual from P_1 is less than the probability of becoming infested by an individual from P_2. Once individuals become infested, they enter P_1,the population of few lice. The diagram of Figure 1 illustrates the flow of individuals in the model.

The model makes use of the following parameters:

μ = per capita rate at which individuals leave the system,

γ_i = per capita treatment rate of P_i individuals,

ω = rate of progression from a state with few lice to a state with many lice, that is, from P_1 to P_2,

c = the average number of contacts per unit of time,

q_i = the probability that a contact with an individual from P_i leads to infestation,

$\beta_i = q_i c$ = the number of successful contacts per unit of time.

To simplify the model, we made several assumptions. We took $\beta_i = q_i c$; that is, we assume that each individual, in the population has the same number of contacts but a different probability of becoming infested. We assume that $\beta_2 > \beta_1$; that is, the number of successful contacts (those that lead to infestation) are greater when an individual comes into contact with a person from P_2 rather than P_1 as we assume that $q_2 > q_1$. Also, we assume that N remains constant. Therefore, we let μN denote the recruitment rate, while μS, μP_1, and μP_2 represent the number of susceptible individuals, individuals infested with few lice, and individuals infested with many lice leaving the system per unit time. Since

$$\mu S + \mu P_1 + \mu P_2 = \mu(S + P_1 + P_2) = \mu N,$$

then N is constant.

2.1 Reduction of the model

Since the population size is constant, we can divide (1), (2) and (3) by N. That, is $\frac{N}{N} = \frac{S}{N} + \frac{P_1}{N} + \frac{P_2}{N}$ where we let $x = \frac{S}{N}$, $y_1 = \frac{P_1}{N}$, $y_2 = \frac{P_2}{N}$, where x is the proportion of susceptibles per unit time; y_1 is the proportion of people with few lice; and y_2 is the proportion of people with many lice. Substituting x, y_1, and y_2 into (1a), (2a), and (3a), we obtain the following equivalent system:

$$\frac{dx}{dt} = \mu - \beta_1 x y_1 - \beta_2 x y_2 - \mu x + \gamma_1 y_1 + \gamma_2 y_2, \tag{5}$$

$$\frac{dy_1}{dt} = \beta_1 x y_1 + \beta_2 x y_2 - (\mu + \gamma_1) y_1 - \omega y_1, \tag{6}$$

$$\frac{dy_2}{dt} = \omega y_1 - (\mu + \gamma_2) y_2. \tag{7}$$

Since $x = 1 - y_1 - y_2$, the system consisting of is reduced to the following two-dimensional systems:

$$\frac{dy_1}{dt} = \beta_1(1 - y_1 - y_2)y_1 + \beta_2(1 - y_1 - y_2)y_2 - (\mu + \gamma_1)y_1 - \omega y_1, \tag{8}$$

$$\frac{dy_2}{dt} = \omega y_1 - \mu y_2 - \gamma_2 y_2. \tag{9}$$

The original system of differential equations reduces to the above simple systems dealing with the proportion of the number of individuals infested with few and many lice.

2.2 Equilibrium points

Endemic equilibrium points are often important if one wishes to find the basic reproductive number R_0. R_0 gives the number of individuals that a single infested person from y_1 or y_2 infests when introduced into a susceptible population. It is crucial that we find R_0, since it is an indication of how quickly the infestation of lice will spread at the beginning of the epidemic. To find R_0, we first find the infestation-free equilibrium point where only susceptible individuals exist in the population.

Setting $\frac{dy_i}{dt} = 0$ $(i = 1, 2)$ gives

$$\beta_1(1 - y_1^* - y_2^*)y_1^* + \beta_2(1 - y_1^* - y_2^*)y_2^* = (\mu + \gamma_1)y_1^* + \omega y_1^* \tag{10}$$

and

$$\omega y_1^* = \mu y_2^* + \gamma_2 y_2^*. \tag{11}$$

Solving for y_1^* from (10), we have that

$$y_1^* = \frac{(\mu + \gamma_2)y_2^*}{\omega}. \tag{12}$$

If we let $A = \frac{\omega}{(\mu+\gamma_2)}$ then $y_2^* = Ay_1^*$. Now we substitute $y_2^* = Ay_1^*$ into (10) to obtain

$$\beta_1(1 - y_1^* - Ay_1^*)y_1^* + \beta_2(1 - y_1^* - Ay_1^*)Ay_1^* = (\mu + \gamma_1)y_1^* + \omega y_1^*. \tag{13}$$

Now we solve for y_1^* and use the fact that $y_2^* = Ay_1^*$ to obtain the endemic equilibrium point:

$$y_1^* = \frac{1}{1 + A}\left(1 - \frac{\mu + \gamma_1 + \omega}{\beta_1 + \beta_2 A}\right), \tag{14}$$

$$y_2^* = \frac{A}{1 + A}\left(1 - \frac{\mu + \gamma_1 + \omega}{\beta_1 + \beta_2 A}\right). \tag{15}$$

For the equilibrium points to be positive (and thus biologically feasible) we need

$$1 - \frac{(\mu + \gamma_1 + \omega)}{\beta_1 + \beta_2 A} > 0$$

which implies that

$$\frac{\mu + \gamma_1 + \omega}{\beta_1 + \beta_2 A} < 1.$$

This condition gives,

$$R_0 = \frac{\beta_1 + \beta_2 A}{\mu + \gamma_1 + \omega}, \tag{16}$$

which can be rewritten as

$$R_0 = \frac{\beta_1}{\mu + \gamma_1 + \omega}\left(1 + \frac{\beta_2}{\beta_1} \cdot \frac{\omega}{\mu + \gamma_2}\right). \tag{17}$$

Now, if $R_0 < 1$, then $(0,0)$ is the unique equilibrium point. If, on the contrary $R_0 > 1$, a unique endemic equilibrium point exists, an equilibrium point that may be expressed as

$$y_1^* = \frac{1}{1+A}\left(1 - \frac{1}{R_0}\right), \tag{18}$$

$$y_2^* = \frac{A}{1+A}\left(1 - \frac{1}{R_0}\right). \tag{19}$$

2.3 Stability of equilibrium points

To find the stability of the equilibrium points, we find the Jacobian matrix of the system (8-9) at its equilibrium points $(0,0)$ and (y_1^*, y_2^*). The general Jacobian matrix is given by

$$J = \begin{pmatrix} -\beta_1 y_1 + \beta_1(1 - y_1 - y_2) - \beta_2 y_2 - (\mu + \gamma_1 + \omega_1) & -\beta_1 y_1 - \beta_2 y_2 + \beta_2(1 - y_1 - y_2) \\ \omega & -(\mu + \gamma_2) \end{pmatrix}.$$

The Jacobian matrix at $(0,0)$ is

$$J(0,0) = \begin{pmatrix} \beta_1 - (\mu + \gamma_1 + \omega) & \beta_2 \\ \omega & -(\mu + \gamma_2) \end{pmatrix}. \tag{20}$$

The equilibrium point (0,0) is stable if the trace is negative and the determinant is positive. Thus, we see that

$$\mathrm{tr}(J(0,0)) = \beta_1 - (2\mu + \gamma_1 + \omega + \gamma_2) < 0$$

iff

$$\frac{\beta_1}{2\mu + \omega + \gamma_1 + \gamma_2} < 1,$$

and

$$\det(J(0,0)) = \beta_1 - (\mu + \gamma_1 + \omega)(-(\mu + \gamma_2)) - \omega\beta_2 > 0$$

iff

$$\frac{\beta_1}{\mu+\gamma_1+\omega}\left(1+\frac{\frac{\omega\beta_2}{\beta_1}}{(\mu+\gamma_1+\omega)(\mu+\gamma_2)}\right)<1.$$

The stability of the infestation-free equilibrium state depends on R_0 being less than 1. To investigate the stability of the endemic equilibrium point (y_1^*, y_2^*) we focus on the general case: that is, the case when $\beta_2 > \beta_1 = \beta$ or equivalently $\beta_2 = q\beta$ where $q > 1$. We let $f(q) = \frac{1+qA}{1+A}$.

The Jacobian matrix in this case is expressed then as

$$\begin{pmatrix} -\beta\left(1-\frac{1}{R_0(q)}\right)f(q)+\frac{\beta}{R_0(q)}-\frac{\beta}{R_0(q)}(1+A)f(q) & -\beta\left(1-\frac{1}{R_0}\right)f(q)+\frac{\beta}{R_0}q \\ \omega & -\frac{\omega}{A} \end{pmatrix}.$$

The trace of J is given by

$$\begin{aligned}\mathrm{tr}(J) &= -\beta\left(1-\frac{1}{R_0(q)}\right)f(q)+\frac{\beta}{R_0(q)}-\frac{\beta}{R_0(q)}(1+A)f(q)-\frac{\omega}{A}\\ &= -\beta(R_0-1)f(q)-\frac{\beta}{R_0}(A)f(q)+\frac{\beta}{R_0}(1-f(q))-\frac{\omega}{A}<0.\end{aligned}$$

This implies that $R_0 > 1$. So, $f(q) > 1$ which implies that $q > 1$.

Now,

$$\begin{aligned}\det(J) &= \left(-\frac{\beta}{R_0}(R_0-1)f(q)+\frac{\beta}{R_0}(1-f(q))-\frac{\beta}{R_0}(A)f(q)\right)\left(-\frac{\omega}{A}\right)\\ &\quad -\omega\left(-\frac{\beta}{R_0}(R_0-1)f(q)+q\frac{\beta}{R_0}\right)\\ &= \frac{\omega}{A}\left(\frac{\beta}{R_0}f(q)-\frac{\beta}{R_0}(1-f(q))+\frac{\beta}{R_0}(A)f(q)\right)+\omega\left(\frac{\beta}{R_0}(R_0-1)f(q)-q\frac{\beta}{R_0}\right)\\ &= (R_0-1)f(q)(1+A)>0\end{aligned}$$

which implies that the endemic equilibria are stable, which implies that $R_0 > 1$. When the condition $R_0 > 1$ is fulfilled, the endemic equilibrium point, (y_1^*, y_2^*) is stable.

2.4 Special case for individuals in Nigeria

We calculated an approximate value for R_0 using figures obtained by Buxton (1938) for males examined at Sokoto, Northern Nigeria in 1938.

Distribution of Lice by Age			
Age	**Total heads**	**No. lice**	**1 to 10 lice**
6 - 10	53	42	5
11 - 15	140	124	14
16 - 20	87	77	7
21 - 30	68	65	3
Total	348	308	29

Table 2: Distribution of lice by age (Buxton 1940)

We will use the formula $R_0 = 1 + \frac{L}{Q}$ by Dietz (1974) to find R_0 using the data from Table 2.4. From the data we calculate that $L = \frac{1}{\mu}$, the time that people are exposed to infestion, is 24 years old, the average age of first infestation is 14.5 years. Thus, when we solve for R_0, we have $R_0 = 2.655$, which means one person with lice will infest 2.655 susceptibles.

The proportion of people with few and many lice from Table 2.4 at equilibrium can be expressed by equations (18) and (19), such that,

$$\begin{aligned} y_1^* &= \frac{1}{1+A}(1 - \frac{1}{R_0}) = \frac{29}{348} \\ y_2^* &= \frac{A}{1+A}(1 - \frac{1}{R_0}) = \frac{319}{348} \\ 1 - \frac{1}{R_0} &= 1 - \frac{1}{2.655} = .623 \end{aligned}$$

By substitution we can calculate A to be 6.5, and we also know that $A = \frac{\omega}{\mu+\gamma_2}$. Now we let $\beta_1 = \beta$ and $\beta_2 = q\beta$, where $q > 1$ and $\beta_2 > \beta_1$, and we substitute the variables into equation (13). Therefore,

$$R_0 = \frac{\beta}{\mu + \omega + \gamma_1} + \frac{qA\beta}{\mu + \omega + \gamma_1}.$$

In this case R_0^f is the contribution to R_0 of individuals infested with few lice and R_0^m is the contribution to R_0 of individuals infested with many lice. That is,

$$R_0^f = \frac{\beta}{\mu + \omega + \gamma_1} \; R_0^m = \frac{qA\beta}{\mu + \omega + \gamma_1} = qAR_0^f.$$

By substitution we have

$$R_0 = R_0^f + R_0^m = R_0^f(1 + qA). \tag{21}$$

We are looking for a condition that will eliminate the epidemic. So we suppose that $R_0^f < 1$, which is when the individuals with few lice do not contribute to the epidemic. To find the condition we do the following:

$$\begin{aligned} R_0^f + R_0^m < 1 &\Leftrightarrow R_0^f(1+qA) < 1 \\ &\Leftrightarrow 1+qA < \frac{1}{R_0^f} \\ &\Leftrightarrow R_0^f < 1 \\ &\Leftrightarrow 1+qA < \frac{\mu+\omega+\gamma_1}{qA\beta} \\ &\Leftrightarrow \gamma_1 > \beta(1+qA) - (\mu+\omega) = \beta qA + \beta - (\mu+\omega) \end{aligned} \tag{22}$$

Hence, equation (22) is the condition that needs to be met to eliminate the epidemic. The condition is shown in Figure 2.

Now we suppose $R_0^f > 1$; that is, individuals with few lice contribute to the epidemic, then

$$R_0 < 1 \text{ iff } R_0^f(1+qA) < 1,$$

which is non-existent. Therefore, the results suggest that one controls the individuals with few lice and if necessary the individuals with many people to have an epidemic free state.

3 Distribution of lice

In summary, the epidemiological model serves to measure the intensity of lice infestation; that is it measures the distribution of lice per human host. The form of this distribution is of great importance to the population dynamics of lice, as well as the distribution between humans. Using the distributions of lice within human communities observed by Buxton in 1940, we can empirically describe the patterns with a negative binomial distribution (NBD).

Observe that it is valid to assume that the distribution of the data shown in Table 3 and Figure 3 can be described as a negative binomial distribution. Notice that if the mean and the variance were to be calculated, the mean would be greater than the variance, which is characteristic of the NBD. In addition, in calculating the skewness we find that it is positive. Our goal is to use the methods of infinite moments to estimate the values of the parameters κ and β. We can later use this information to make predictions about the lice distribution in a closed population. Also, we want to fit the data using the method of goodness of fit to approximate the behavior of the distribution.

Distribution of lice in a tropical jail	
Number of lice	Number of individuals
1 to 2	49
3 to 10	32
11 to 25	22
26 to 100	13
101 and up	9

Table 3: Distribution of lice in 125 infested individuals in a tropical jail (Buxton 1940).

This model consists of two differential equations, where n_0, n_i represent the population size of individuals with zero lice and the population of individuals with i lice, respectively. We establish that the total population is constant. The model is as follows:

$$\frac{dn0}{dt} = -(\mu + \psi)n_0 + \sigma n_1 + \Lambda \tag{23}$$

$$\frac{dni}{dt} = -(\mu + \psi + i\sigma)n_i + \sigma(i+1)n_{i+1} + \psi n_{i-1} \tag{24}$$

where $\frac{dn0}{dt}$, $\frac{dni}{dt}$ are the population rates without and with i lice, respectively.

We have that the total population at time t is given by $N(t)$is a constant N. Further, the total population with i lice at time t is $P(t)$.

$$\begin{aligned} N(t) &= \sum_{i=0}^{\infty} n_i = N, \\ P(t) &= \sum_{i=0}^{\infty} i n_i, \\ \Pi(t) &= \omega = \text{ second moment}, \\ \Delta(t) &= \delta = \text{ third moment}. \end{aligned}$$

The parameters are defined as follows:

$\frac{1}{\mu}$ = average time in which the people are in the system,

σ = mortality rate of the lice,

$\psi(k, \beta)$ = rate of infestation (to be specified),

$\Lambda = 3N(t) =$ rate of entrance into the population,

$k =$ clumping parameter of lice.

Observe that we have an infinite number of equations. Now, calculate the moments to fit our data (Table 3) to a negative binomial distribution. So,

$$\begin{aligned}
N'(t) &= \sum_{i=0}^{\infty} \frac{dni}{dt} \\
&= -\sum_{i=0}^{\infty}(\mu+\psi+i\sigma)n_i + \sum_{i=0}^{\infty}\sigma(i+1)n_i + \sum_{i=0}^{\infty}\psi n_{i-1} + \Lambda \\
&= \Lambda - \mu N \qquad (25) \\
P'(t) &= \sum_{i=0}^{\infty} i\frac{dn_i}{dt}(t) \\
&= -\sum_{i=0}^{\infty}(\mu+\psi+i\sigma)in_i + \sum_{i=1}^{\infty}\sigma(i+1)in_i + \sum_{i=1}^{\infty}\psi i n_{i-1} \\
&= \psi N - P(\mu+\sigma) \qquad (26) \\
\Pi'(t) &= -\sum_{i=0}^{\infty}(\mu+\psi+i\sigma)i^2n_i + \sum_{i=1}^{\infty}\sigma(i+1)i^2n_i + \sum_{i=1}^{\infty}\psi i^2 n_{i-1} \\
&= -\Pi(t)(\mu+2\sigma) + P(t)(2\psi+\sigma) + \psi N(t) \qquad (27) \\
\Delta'(t) &= -\sum_{i=0}^{\infty}(\mu+\psi+i\sigma)i^3n_i + \sum_{i=1}^{\infty}\sigma(i+1)i^3n_i + \sum_{i=1}^{\infty}\psi i^3 n_{i-1} \\
&= \psi(3\theta+3P(t)+N(t)) - \mu\Delta(t) - \sigma(3\theta+3\Delta(t)) - P(t). \qquad (28)
\end{aligned}$$

Note that each equation has a function, $\psi(t)$, associated with it.

We now proceed to look for the equilibrium points. To do so, we set (25-28) to zero and find the following equilibria:

$$\begin{aligned}
N &= \frac{\Delta}{\mu} \\
P(t) &= \frac{\psi N}{\mu+\sigma} \\
\theta(t) &= \frac{P(t)(2\psi+\sigma)+\psi N}{\mu+2\sigma}.
\end{aligned}$$

In addition, we need to divide by N in equations P, and θ. This step gives

$$\frac{P(t)}{N} = \frac{\psi}{\mu + \sigma} = m. \tag{29}$$

Note that m denotes the mean. In addition, notice that $P(t)/N$ is the total population with i lice divided by the total population. Also, we find that

$$\frac{\theta(t)}{N} = \frac{\frac{P(t)}{N}(2\psi + \sigma) + \psi}{\mu + 2\sigma} = \frac{m(2\psi + \sigma) + \psi}{\mu + 2\sigma} = \omega, \tag{30}$$

where ω denotes the second moment over N.

To find the variance we need to calculate: Variance = ω - m^2. The challenge of our model is find a function ψ that will serve to close our moments.

4 Results

After trying several functions of $\psi(k, \beta)$ we were unable to estimate the values of k and β. We always obtained systems of two parallel lines. Although the result for the mean, variance, and the third moment were consistently positive, even with the characteristics of the negative binomial distribution, we were unable to fit $\psi(k, \beta)$ to our data. We believe that the problem arises from the fact that our data is measured at a specific time. However, the distribution changes with time. We also tried to adjust the negative binomial using the goodness of fitness method. The results for this method were that $k = 22, \beta = .9$, and the sum of the squares = 1058.7, but these results were inconsistent with the observed data. The reason for this inconsistency is due to the disorderliness of the data. Also, we do not have access to the data so we cannot make an arrangement in the distribution.

We have several suggestions about how to resolve the problem of moments closure. For instance, we need data that change with time. Perhaps we can make data measurements each month and then calculate the mean and variance for each one. We can then construct a distribution that changes with time. Finally, we will then be able to construct a system of equations that will serve to estimate the values for k and β.

5 Conclusions and directions for future work

The results of the epidemiological model indicate that in order to prevent an epidemic of lice, those individuals infested with few lice must be treated at least to the point where $R_0 > 1$. Also, if $R_0 < 1$, we know that with time, the number of lice-infested individuals approaches zero. That is, there is no epidemic. On the other hand, if $R_0 > 1$, the lice epidemic persists. The model of the distribution of head lice in humans gave relevant results. The differential equation model with appropriate infection rates ψ indicated that the mean is less than the variance. We also know that the skewness is positive. In addition, if ψ is given by a parameter curve, $\upsilon = \psi(k, \beta)$, then longitudinal data is needed to estimate k and β. It is not enough to have data that represent an instant. We need to obtain more dispersed data.

It is important to continue our work in some way because lice infestation is a problem that affects almost all the children in our schools. If we know more about lice distribution, we can make more effective controls, which will result in a minimum propagation.

Acknowledgments

This study was supported by the following institutions and grants: National Science Foundation (NSF Grant DMS-9600027); National Security Agency (NSA Grants MDA 904-96-1-0032 and MDA 904-97-1-0074); Presidential Faculty Fellowship Award (NSF Grant DEB 925370) and Presidential Mentoring Award (NSF Grant HRD 9724850) to Carlos Castillo-Chavez; and the Office of the Provost of Cornell University. The authors are solely responsible for the views and opinions expressed in this report. The research in this report does not necessarily reflect the views and/or opinions of the funding agencies and/or Cornell University.

We would like to sincerely thank the following individuals for their excellent advice, support, and guidance throughout our project: Jorge Velasco-Hernandez, Carlos M. Hernandez, Steve Tennenbaum, Ariel Rodríguez, Pamela Hammons, Phil Acree Cavalier, Denise E. Kirschner, Aníbal Muñoz, and David Sanchez. Special thanks to Carlos Castillo-Chavez for his patience, dedication, and wisdom. We offer our utmost gratitude to Julio Villarreal for providing persistent encouragement and support, as well as being a true facilitator of knowledge.

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A Appendix

A.1 Reduction of the Model

We have that

$$\begin{aligned}
\frac{dx}{dt} &= \mu - \beta_1 x y_1 - \beta_2 x y_2 - \mu x + \gamma_1 y_1 + \gamma_2 y_2 \\
\frac{dy_1}{dt} &= \beta_1 x y_1 + \beta_2 x y_2 - (\mu + \gamma_1) y_1 - \omega y_1 \\
\frac{dy_2}{dt} &= \omega y_1 - (\mu + \gamma_2) y_2.
\end{aligned}$$

Using the fact that $x = 1 + y_1 + y_2$ through substitution, we reduce the number of equations to obtain

$$\frac{dy_1}{dt} = \beta_1(1 - y_1 - y_2)y_1 + \beta_2(1 - y_1 - y_2)y_2 - (\mu + \gamma_1)y_1 - \omega y_1 \quad (31)$$

$$\frac{dy_2}{dt} = \omega y_1 - \mu y_2 - \gamma_2 y_2. \quad (32)$$

A.2 Finding R_0.

To find R_0, we look for feasibility conditions that allow the existence of an endemic equilibrium point. We do this as follows:

We first find the equilibrium points. Setting $\frac{dy_1}{dt} = 0$ gives

$$\beta_1(1 - y_1 - y_2)y_1 + \beta_2(1 - y_1 - y_2)y_2 = (\mu + \gamma_1)y_1 + \omega y_1. \tag{33}$$

When $\frac{dy_2}{dt} = 0$ we have

$$\omega y_1 = \mu y_2 + \gamma_2 y_2. \tag{34}$$

Solving for y_1, we have

$$y_1 = \frac{(\mu + \gamma_2)y_2}{\omega}. \tag{35}$$

Now let $A = \frac{\omega}{\mu+\gamma_2}$, then $y_2 = Ay_1$. Now we substitute $y_2 = Ay_1$ into (33) to obtain

$$\beta_1(1 - y_1 - Ay_1)y_1 + \beta_2(1 - y_1 - Ay_1)Ay_1 = (\mu + \gamma_1)y_1 + \omega y_1. \tag{36}$$

Now we solve for y_1 and then use the fact that $y_2 = Ay_1$ to obtain the endemic equilibrium point:

$$y_1^* = \frac{1}{1+A}\left(1 - \frac{\mu + \gamma_1 + \omega}{\beta_1 + \beta_2 A}\right), \tag{37}$$

$$y_2^* = \frac{A}{1+A}\left(1 - \frac{\mu + \gamma_1 + \omega}{\beta_1 + \beta_2 A}\right). \tag{38}$$

For the equilibrium points to be positive (and thus biologically feasible) we need $1 - \frac{\mu+\gamma_1+\omega}{\beta_1+\beta_2 A} > 0$, which implies that $\frac{\mu+\gamma_1+\omega}{\beta_1+\beta_2 A} < 1$. Notice that this condition is equivalent to saying that $\frac{\beta_1+\beta_2 A}{\mu+\gamma_1+\omega} < 1$. This condition implies that R_0 can be defined as

$$R_0 = \frac{\beta_1 + \beta_2 A}{\mu + \gamma_1 + \omega}.$$

Note that R_0 can be rewritten as

$$R_0 = \frac{\beta_1}{\mu + \gamma_1 + \omega} + \left(\frac{\beta_2}{\mu + \gamma_1 + \omega}\right)\left(\frac{\omega}{\mu + \gamma_2}\right).$$

Then, obviously if $R_0 < 1$, then $(0, 0)$ is the unique equilibrium point. If, on the contrary $R_0 > 1$, the endemic equilibrium point exists.

A.3 Discussion of the Stability of the Equilibrium Points

To find the stability of the equilibrium points, we need to find the Jacobian matrix of our system:

$$\begin{aligned}\frac{dy_1}{dt} &= \beta_1(1-y_1-y_2)y_1+\beta_2(1-y_1-y_2)y_2-(\mu+\gamma_1)y_1-\omega y_1,\\ \frac{dy_2}{dt} &= \omega y_1-\mu y_2-\gamma_2 y_2.\end{aligned}$$

where the equilibrium points are $(0,0)$ and (y_1^*, y_2^*). The Jacobian matrix is,

$$J=\begin{pmatrix} -\beta_1 y_1+\beta_1(1-y_1-y_2)-\beta_2 y_2-(\mu+\gamma_1+\omega_1) & -\beta_1 y_1-\beta_2 y_2+\beta_2(1-y_1-y_2) \\ \omega & -(\mu+\gamma_2)\end{pmatrix}.$$

We now evaluate the equilibrium points $(0,0)$ in the Jacobian matrix,

$$J(0,0)=\begin{pmatrix}\beta_1-(\mu+\gamma_1+\omega) & \beta_2 \\ \omega & -(\mu+\gamma_2)\end{pmatrix}.$$

In order to show that the equilibrium point $(0,0)$ is stable we need to show that the trace is negative and the determinant is positive. Thus, we see that

$$\mathrm{tr}J(0,0)=\beta_1-(2\mu+\gamma_1+\omega+\gamma_2)<0,$$

iff

$$\frac{\beta_1}{2\mu+\omega+\gamma_1+\gamma_2}<1,$$

and

$$\det J(0,0)=(\beta_1-(\mu+\gamma_1+\omega))(-(\mu+\gamma_2))-\omega\beta_2<0,$$

iff

$$R_0=\frac{\beta_1}{\mu+\gamma_1+\omega}+\frac{\omega\beta_2}{(\mu+\gamma_1+\omega)(\mu+\gamma_2)}<1.$$

Obviously, if $R_0<1$ then

$$\frac{\beta_1}{2\mu+\omega+\gamma_1+\gamma_2}<1.$$

To investigate the stability of the endemic equilibrium point, (y_1^*, y_2^*), we study two cases:

Case 1: $\beta_1=\beta_2=\beta$.

We take $\frac{\partial y_1}{\partial y_1}$ to simplify the Jacobian matrix, J:

$$\frac{\partial y_1}{\partial y_1} = -\beta_1 y_1 + \beta_1(1 - y_1 - y_2) - \beta_2 y_2 - (\mu + \gamma_1 + \omega),$$

since $\beta_1 = \beta_2 = \beta$ then

$$\frac{\partial y_1}{\partial y_1} = -\beta y_1 + \beta(1 - y_1 - y_2) - \beta y_2 - (\mu + \gamma_1 + \omega);$$

$$R_0 = \frac{\beta}{\mu + \gamma_1 + \omega)}(1 + A).$$

The Jacobian matrix takes on the following form:

$$\begin{pmatrix} -\beta\left(1 - \frac{1}{R_0}\right) + \frac{\beta}{R_0} - \frac{\beta}{R_0}(1 + A) & -\beta\left(1 - \frac{1}{R_0}\right) + \frac{\beta}{R_0} \\ \omega & -\frac{\omega}{A} \end{pmatrix}.$$

We now find the trace and the determinant:

$$\begin{aligned} \mathrm{tr}(J) &= -\beta\left(1 - \frac{1}{R_0}\right) + \frac{\beta}{R_0} - \frac{\beta}{R_0}(1 + A) - \frac{\omega}{A} \\ &= -\beta\left(R_0 - 1\right) - \frac{\beta}{R_0}A - \frac{\omega}{A} < 0. \end{aligned}$$

Notice that in order for the trace to be negative, we need $R_0 - 1 > 0$ which is the equivalent to saying that $R_0 > 1$.

$$\begin{aligned} \det(J) &= \left(-\beta\left(1 - \frac{1}{R_0}\right) + \frac{\beta}{R_0} - \frac{\beta}{R_0}(1 + A)\right)\left(-\frac{\omega}{A}\right) - \omega\left(-\beta\left(1 - \frac{1}{R_0}\right) + \frac{\beta}{R_0}\right) \\ &= \frac{\beta}{R_0}(R_0 - 1) + \frac{\beta}{R_0}A + A\left(\frac{\beta}{R_0}(R_0 - 1) - \frac{\beta}{R_0}\right) \\ &= \frac{\beta}{R_0}(1 + A)(R_0 - 1) > 0. \end{aligned}$$

Similarly, for the determinant to be positive, we need $R_0 - 1 > 0$, which is equivalent to saying that $R_0 > 1$.

For this case, $R_0 > 1$ shows that the endemic equilibrium point is stable when $\beta_1 = \beta_2 = \beta$.

Case 2: $\beta_2 > \beta_1 = \beta$ such that $\beta_2 = q\beta$ where $q > 1$.

We now take $\frac{\partial y_1}{\partial y_1}$ to simplify the Jacobian matrix. Using the fact that $\beta_2 > \beta_1 = \beta$ such that $\beta_2 = q\beta$ where $q > 1$, substitution gives

$$\begin{aligned}\frac{\partial y_1}{\partial y_1} &= -betay_1 + \beta(1 - y_1 - y_2) - q\beta y_2 - (\mu + \gamma_1 + \omega) \\ &= -\beta y_1 - q\beta y_2 + \beta(1 - y_1 - y_2) - (\mu + \gamma_1 + \omega).\end{aligned}$$

Since we want the Jacobian matrix in terms of R_0, we simplify (9) by taking it apart.

Simplifying $-\beta y_1 - q\beta y_2 = -\beta(y_1 - qy_2)$. Now, substituting (y_1^*, y_2^*) from (8*a*) and (8*b*) gives the following expression,

$$-\beta\left(\left(1 - \frac{1}{R_0(q)}\right)\frac{1}{1+A} + \left(1 - \frac{1}{R_0(q)}\right)\frac{qA}{1+A}\right) = -\beta\left(1 - \frac{1}{R_0(q)}\right)\left(\frac{1+qA}{1+A}\right)$$

where $q > 1$. Now, substituting $f(q) = \frac{1+qA}{1+A}$ into the expression above gives

$$-\beta\left(1 - \frac{1}{R_0(q)}\right) f(q).$$

where evaluating $f(1) = 1$, we have Case 1, where $q > 1$.

Now, simplifying $\beta(1 - y_1 - y_2)$ we have:

$$\beta_1(1 - y_1 - y_2) = \beta(1 - (y_1 + y_2)) = \beta\left(1 - \left(1 - \frac{1}{R_0(q)}\right)\right) = \frac{\beta}{R_0(q)}$$

and

$$\beta_2(1 - y_1 - y_2) = q\beta(1 - (y_1 + y_2)) = q\beta\left(1 - \left(1 - \frac{1}{R_0(q)}\right)\right) = \frac{q\beta}{R_0(q)}.$$

So, putting (9a) and (9b) together gives

$$\begin{aligned}\frac{\partial y_1}{\partial y_1} &= -\beta\left(1 - \frac{1}{R_0(q)}\right)\frac{1+qA}{1+A} + \frac{\beta}{R_0(q)} - \frac{\beta}{R_0(q)}(1+A)\frac{qA}{1+A} \\ &= -\beta\left(1 - \frac{1}{R_0(q)}\right) f(q) + \frac{\beta}{R_0(q)} - \frac{\beta}{R_0(q)}(1+A)f(q)\end{aligned}$$

where $f(q) = \frac{1+qA}{1+A}$.

$$R_0(q) = \frac{\beta_1}{\mu + \gamma_1 + \omega} + \frac{\beta_2}{\mu + \gamma_1 + \omega} \cdot \frac{\omega}{\mu + \gamma_2} = \frac{\beta_1}{\mu + \gamma_1 + \omega} + \frac{q\beta}{\mu + \gamma_1 + \omega} A,$$

since $A = \frac{\omega}{\mu+\gamma_2}$, we have

$$R_0(q) = \frac{\beta}{\mu+\gamma_1+\omega}(1+qA).$$

We now have that the Jacobian matrix is of the form:

$$\begin{pmatrix} -\beta\left(1-\frac{1}{R_0(q)}\right)f(q) + \frac{\beta}{R_0(q)} - \frac{\beta}{R_0(q)}(1+A)f(q) & -\beta\left(1-\frac{1}{R_0}\right)f(q) + \frac{\beta}{R_0}q \\ \omega & -\frac{\omega}{A} \end{pmatrix}.$$

We now find the trace and the determinant of J:

$$\begin{aligned} \mathrm{tr}(J) &= -\beta\left(1-\frac{1}{R_0(q)}\right)f(q) + \frac{\beta}{R_0(q)} - \frac{\beta}{R_0(q)}(1+A)f(q) - \frac{\omega}{A} \\ &= -\beta(R_0-1)f(q) - \frac{\beta}{R_0}Af(q) + \frac{\beta}{R_0}(1-f(q)) - \frac{\omega}{A} < 0. \end{aligned}$$

This implies that $R_0 > 1$. So, $f(q) > 1$, which implies that $q > 1$. Now,

$$\begin{aligned} \det(J) &= \left(-\frac{\beta}{R_0}(R_0-1)f(q) + \frac{\beta}{R_0}(1-f(q)) - \frac{\beta}{R_0}Af(q)\right)\left(-\frac{\omega}{A}\right) \\ &\quad -\omega\left(-\frac{\beta}{R_0}(R_0-1)f(q) + q\frac{\beta}{R_0}\right) \\ &= \frac{\omega}{A}\left(\frac{\beta}{R_0}cf(q) - \frac{\beta}{R_0}(1-f(q)) + \frac{\beta}{R_0}Af(q)\right) + \omega\left(\frac{\beta}{R_0}(R_0-1)f(q) - q\frac{\beta}{R_0}\right) \\ &= (R_0-1)f(q)(1+A) > 0 \end{aligned}$$

which implies that $R_0 > 1$.

When the condition $R_0 > 1$ is fulfilled, the endemic equilibrium point, (y_1^*, y_2^*) is stable when $\beta_2 > \beta_1 = \beta$.

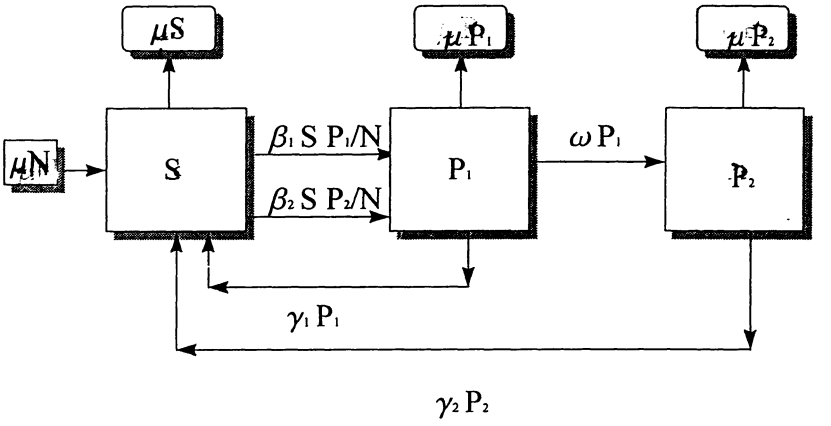

Figure 1: Compartmental model.

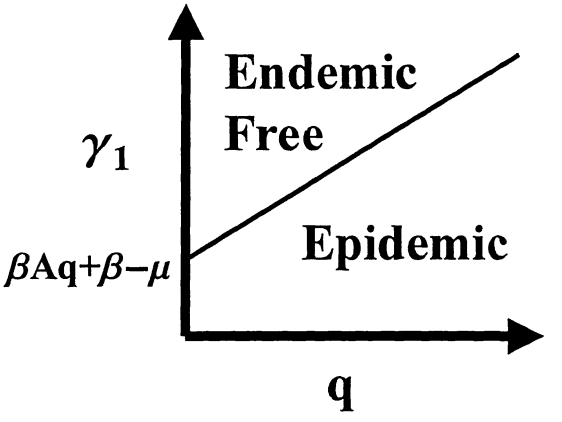

Figure 2: Bifurcation diagram in the parameters q and γ_1.

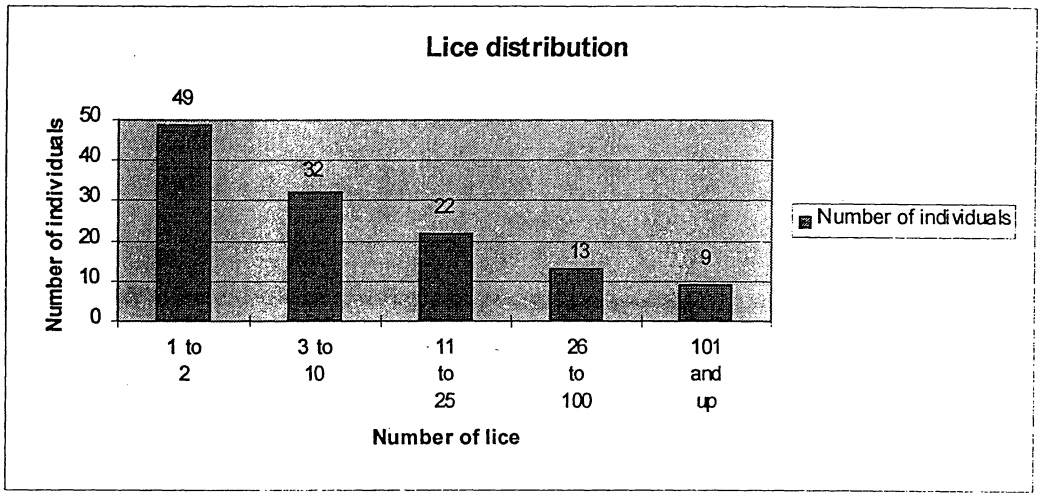

Figure 3: Lice distribution (Buxton 1940).