A Competition Model for Advertised Companies

Victor A. Chacón
Occidental College
Patricia Fuentes
Loyola Marymount University
George F. González
Rice University
Mario A. Mendieta
Universidad Autonoma Metropolitana, Iztapalapa

1 Introduction

The idea of two-species competition in mathematical population biology was first introduced by Lotka and Volterra in the 1920s. They proposed "that a population of competitors finds less of the same resources and cannot grow to its maximum capacity". In this situation species compete for the same, limited resources. In advertising, competing companies find themselves in comparable situations. For example, Budweiser and Coors Light are two of the top three selling brands of beer in the United States. The companies that produce Budweiser and Coors Light beers spend millions of dollars each year on television advertising to entice the 80 million beer drinkers that exist in the United States to purchase their particular brands. Thus these two brands are competing for the same, limited resource: beer drinkers.

Of the various types of media advertising, television is probably the most widely used and influential in attracting consumers. In the last four years, beer companies have spent over \$700 million per year on television advertising. In fact, in 1994 Budweiser and Coors Light spent \$89 million and \$58 million, respectively (U.S. Beer Market 307). In addition eighty million Americans, about 1/3 of the population, are beer drinkers. Also, rarely a person exists in the United States who does not watch television. Therefore, it is interesting to study the impact that beer commercials have on the beer drinking population that watches television since this population makes up such a large percentage of American society. Analyzing this situation helps one understand the competition between two leading brands in their attempts to increase their share of the market.

In this project we use a deterministic mathematical model to study the dynamical behavior of a system in which two brands of beer compete. In order to understand the competition between Budweiser and Coors Light, we treat this situation as a Susceptible-Infected-Susceptible epidemiological model. In this case, the susceptible population consists of beer drinkers who watch television and are at least 21 years of age but do not drink Budweiser or Coors Light. The infected populations are those people that purchase either Budweiser or Coors Light. This model takes into account the number of commercials shown for each brand per unit of time. At this point we need to point out a couple of assumptions we made. First, we assume that people purchase either Budweiser or Coors Light, but not both. In addition, people purchase the beers solely on the basis of watching television commercials. So in analyzing the deterministic model, we ascertain an economic aspect of the competition between the two beers; that is, we can see how the amount of money that beer companies spend on television advertising in turn influences an increase in their sales.

In the second model that we will present in this report, the companies are competing for both beer consumers and television air time. This model is just a slight variation of our original SIS model. Furthermore, if one company buys many commercial slots, the other company is limited in that it cannot air as many commercials as it can afford to broadcast. Therefore, we will show that the company with fewer commercials does not attract as many beer consumers as it otherwise would.

Due to the fact that real life is not deterministic, we introduce variability into the model. By running a computer simulation that takes variability into account, we will analyze stochastic effects. We will also discuss the result of the simulation in the latter part of this report.

Even though this model is formulated to analyze the competition between Budweiser and Coors Light, we can generalize it for any two competing products. Of course the results would not be the same in every case because the values of the parameters will change, and different products will target distinct types of consumers.

In this project we hope to demonstrate the relationship between television advertising and the purchasing behavior of the susceptible population in choosing either Budweiser or Coors Light. Also, we hope to demonstrate that at some point the influence of television advertising can allow certain economic situations to occur. In one situation television advertisements are no longer broadcast. Therefore, members of the susceptible population no longer purchase either Budweiser or Coors Light beer products based on watching television advertisements. In turn, both Budweiser and Coors Light leave the television advertising market. On the other hand, under certain circumstances one of the two beer companies succeeds in "infecting" many more susceptibles than the other. Therefore, one company has no sales as a result of its television advertisements. Finally, for certain conditions we find that both companies continue to stay in business, (*i. e.* they both have a sufficient number of consumers buying their particular brands of beer to keep them in business). Moreover, we may be able to show that at some point the #3 beer, Coors Light, may become the top-selling beer.

2 Deterministic Models

We are interested in investigating the effects that television advertising has on the beer drinking population in terms of purchasing behavior. Since our investigation depends on television beer commercials, we chose two high-selling brands, Budweiser and Coors Light, because they are two beer companies that can afford to spend a great deal more money on television advertising than smaller beer companies.

In our study beer purchasing is considered as an epidemic. First, when a person watches a television commercial which advertises either Budweiser or Coors Light, we say that this is a contact. Therefore, as the number of commercials aired increases, so does the number of contacts. This increase in the number of contacts per unit of time is called the contact rate. As a result, the number of people who purchase beer increases rapidly. This is the same type of behavior that occurs during an epidemic.

We have learned that one way to mathematically model an epidemic is by using the SIS (Susceptible-Infected-Susceptible) model. The SIS model is a variation of the classical SIR (Susceptible-Infected-Recovered) model studied by Kermack and McKendrick. In the general SIS model, recovery is not considered. Once a person enters the infected state, he or she either returns to the susceptible population or leaves the system entirely. Thus this model proves very useful in our situation since once a person purchases Budweiser or Coors Light, that person returns to the susceptible population or leaves the system for good.

In the first part of our project we studied the deterministic form of our model. The five equations below model our situation. They are as follows:

$$\frac{dS}{dt} = \mu N - \beta_1\left(\frac{ST_1}{T_1+T_2+\alpha}\right) - \beta_2\left(\frac{ST_2}{T_1+T_2+\alpha}\right) - \mu S + \gamma_1 I_1 + \gamma_2 I_2 \qquad (1)$$

$$\frac{dI_1}{dt} = \beta_1\left(\frac{ST_1}{T_1+T_2+\alpha}\right) - (\gamma_1+\mu)I_1 \qquad (2)$$

$$\frac{dI_2}{dt} = \beta_2\left(\frac{ST_2}{T_1+T_2+\alpha}\right) - (\gamma_2+\mu)I_2 \tag{3}$$

$$\frac{dT_1}{dt} = r_1\left(1-\frac{I_1}{I_1+I_2+\omega}\right)\left(1-\frac{T_1}{k_1}\right)T_1 \tag{4}$$

$$\frac{dT_2}{dt} = r_2\left(1-\frac{I_2}{I_1+I_2+\omega}\right)\left(1-\frac{T_2}{k_2}\right)T_2. \tag{5}$$

Now a list of our parameters and variables and a brief description of them is in order.

$S(t) =$ susceptibles; the number of beer drinkers who watch television commercials and are at least 21 years of age at time t;

$I_1(t) =$ the number of people who purchase Budweiser beer after having watched television commercials at time t;

$I_2(t) =$ the number of people who purchase Coors Light beer after having watched television commercials at time t;

$T_1(t) =$ the number of Budweiser commercials at time t;

$T_2(t) =$ the number of Coors Light commercials at time t;

$\mu =$ death rate, *i. e.*, the rate at which people leave the system for whatever reason;

$N =$ total beer drinking population: $N = S + I_1 + I_2$;

$\beta_1 =$ also defined as cP_1, where c is the average number of commercials watched per unit of time t, and P_1 is the probability of purchasing Budweiser beer after watching television advertising for this brand;

$\beta_2 =$ also defined as cP_2, where c is the average number of commercials watched per unit of time t, and P_2 is the probability of purchasing Coors Light beer after watching television advertising for this brand;

$\alpha =$ delay time before purchasing either Budweiser or Coors Light beer;

$\gamma_i =$ the rate of return from infected population back to susceptible population; where $i = 1$ for Budweiser and $i = 2$ for Coors Light;

$\omega =$ numerical value between 0 and 1 that keeps the denominator from equaling zero;

$k_1 =$ the maximum number of commercials that Budweiser can afford per unit of time;

$k_2 =$ the maximum number of commercials that Coors Light can afford per unit of time.

The compartmental diagram of Figure 1 represent our system of equations. In this case S, the susceptible population, is the beer drinking population that watches television commercials and is at least twenty-one years old. The infected population can be described as I_1 and I_2, where I_1 is the beer drinking population that purchases Budweiser after coming in contact with television commercials for this brand and I_2 is the number of susceptibles who purchase Coors Light after having watched television commercials for this brand. Once a susceptible is infected, one of two events happen: a person can return to the susceptible population when he/she watches television commercials again or an infected person can leave the system entirely, (i.e. they die, develop cirrhosis or they just do not drink beer any longer). In order to describe how the populations S, I_1, and I_2 change with respect to the parameters, we analyze the differential equations that are part of our SIS model.

We reduce the number of equations to four for simplicity purposes. We noticed that since

$$\frac{dN}{dt} = \frac{dS}{dt} + \frac{dI_1}{dt} + \frac{dI_2}{dt} = 0,$$

then the value of N is just a constant. Further, we found that since dS/dt is a combination of dI_1/dt and dI_2/dt we could have just worked with equations (2-5). We wanted to find the equilibrium points of the system because these points give us a better understanding of the local dynamics. The first step in the process was to set $dT_1/dt = 0$ and solve for T_1. We found that $T_1 = 0$ or $T_1 = k_1$. We also did this for dT_2/dt to solve for T_2. The values that we found for T_2 were $T_2 = 0$ or $T_2 = k_2$. At this point, we began to use the program Mathematica to help us determine the remaining values, I_1 and I_2 for the equilibrium points. The four equilibrium points are referred to as E_0, E_1, E_2, E_3 respectively (note that $S = N - I_1 - I_2$). They are as follows:

$$\begin{aligned}
E_0 &= (N, 0, 0, 0, 0), \\
E_1 &= (S, (\beta_1 k_1 N)(\alpha\gamma_1 + \beta_1 k_1 + \gamma_1 k_1 + \alpha\mu + k_1\mu), 0, k_1, 0), \\
E_2 &= (S, 0, (\beta_2 k_2 N)(\alpha\gamma_2 + \beta_2 k_2 + \gamma_2 k_2 + \alpha\mu + k_2\mu), 0, k_2), \\
E_3 &= \Bigg(S, \frac{\beta_1 k_1(\gamma_2+\mu)N}{\beta_1 k_1(\gamma_2+\mu) + \alpha(\gamma_1+\mu)(\gamma_2+\mu) + (\gamma_1+\mu)(\beta_1 k_1 + (k_1+k_2)(\gamma_2+\mu))}, \\
&\quad \frac{\beta_2 k_2(\gamma_1+\mu)N}{\beta_1 k_1(\gamma_2+\mu) + \alpha(\gamma_1+\mu)(\gamma_2+\mu) + (\gamma_1+\mu)(\beta_2 k_2 + (k_1+k_2)(\gamma_2+\mu))}, k_1, k_2\Bigg).
\end{aligned}$$

The next step of the process was to determine the stability of our equilibrium points. We did this by first determining the Jacobian Matrix of the four differential equations and then evaluating it at each of the four equilibrium points. This gives us the eigenvalues for each of the equilibrium points. If each of the eigenvalues is negative, then this implies that the equilibrium point is stable. If there is a combination of positive and negative eigenvalues, then the equilibrium point is said to be semi-stable. If all the eigenvalues are positive, then the equilibrium point is unstable.

Before we proceed notice that E_0 is the disease free state since $I_1 = I_2 = T_1 = T_2 = 0$ for this case. This means that at this point no commercials are being shown for either brand of beer and thus no infected persons are purchasing beer. For E_0, we found that since there is a combination of positive and negative eigenvalues that this equilibrium point is semi-stable.

Next, we interpret the meaning of E_1. Again, as for E_0 there exists a combination of positive and negative eigenvalues. This implies that E_1 is also a semi-stable equilibrium point. Now, one notices that at this point $I_1 > 0$ and $I_2 = 0$. This means that Coors Light is not selling any beer at all since $T_2 = 0$, (*i. e.* the company is not producing any television commercials at time t). In addition, Budweiser has no competition with Coors Light.

E_2 is a complement of E_1. In other words since $I_1 = 0$ and $I_2 > 0$, then Budweiser is not selling beer. Thus, Coors Light has no competition with Budweiser.

E_3 is the equilibrium point for the endemic state. Since all of the eigenvalues for this equilibrium point are negative, we can conclude that E_3 is stable (see appendix). In fact, of the four equilibrium points E_3 is the only one that is stable. At this point we find that people are purchasing beer from both Budweiser and Coors Light. Something else that we found interesting was that all values for T_1, T_2, I_1, I_2 were nonnegative for the different equilibrium points. Therefore, all the equilibrium points always exist (see appendix).

After evaluating our equilibrium points and finding their stability status, we proceeded to estimate the values for our parameters. First, we found that the average cost to run a television commercial is about \$75,000 for thirty seconds and that there are 80 million beer drinkers in the United States (Budweiser 1997). In our research we found the following values that helped us estimate our parameters:

Total spent on TV ads by Budweiser	\$ 88,947,000
Total spent on TV ads by Coors Light	\$ 57,615,000.
	(U.S. Beer Market 1995)

Calculating k_1 *and* k_2*:*

$$
\begin{aligned}
k_1 \text{ (value for Budweiser)} &= \text{total spent on TV ads by Bud / avg cost to run TV commercial} \\
&= \$88{,}947{,}000 \,/\, \$75{,}000 \\
&= 1200 \text{ commercials per year} = 100 \text{ commercials per month} \\
&= 1200/365 \text{ per day.} \\
k_2 \text{ (value for Coors Light)} &= \text{total spent on TV ads by Coors/ avg cost to run TV commercial} \\
&= \$57{,}615{,}000 \,/\, \$75{,}000 \\
&= 768 \text{ commercials per year} = 64 \text{ commercials per month} \\
&= 768/365 \text{ per day.}
\end{aligned}
$$

Estimating P1 and P2:

Company	TV Advertising	Market Share
Coors Light	96.8%	6.6%
Budweiser	89.1%	20.9%

Now,

$$
\begin{aligned}
P_1 &= 0.891(.209) = .186 \\
P_2 &= 0.968(.066) = .064
\end{aligned}
$$

Estimating μ, γ_1, γ_2, α and ω:

We decided that the average time that people spend as beer drinkers is about 45 years. That is, a person lives about 45 years on average from the time he becomes a beer drinker to the time he dies. In days this rate equals $1/(45 \cdot 365)$ which is the value for μ. Then we assumed that one week elapses between the times that the average person buys beer. So this rate equates to $1/7$. We assigned this value to both γ_1 and γ_2. For our model, α and ω are arbitrary parameters. These parameters keep our denominator from equaling zero. Therefore, we decided to make them both equal to 1 for simplicity.

We modified our model slightly by adding competition for television air time. That is, both brands are competing for beer drinkers as well as the opportunity to broadcast their commercials. The equations remained the same as in the original model except for the last two. The revised equations are as follows:

$$
\frac{dT_1}{dt} = r_1\left(1 - \frac{I_1}{I_1 + I_2 + \omega}\right)\left(1 - \frac{T_1 + \delta_2 T_2}{k_1}\right)T_1, \tag{6}
$$

$$\frac{dT_2}{dt} = r_2\left(1-\frac{I_2}{I_1+I_2+\omega}\right)\left(1-\frac{T_2+\delta_1 T_1}{k_2}\right)T_2. \tag{7}$$

We followed the same procedure stated earlier in order the calculate the Jacobian and the equilibrium points for this particular version of the model. Two new parameters, δ_1 and δ_2, are introduced. Here δ_1 is the rate at which T_1 decreases the number of commercials for T_2. Likewise, δ_2 is the rate at which T_2 decreases the number of commercials for T_1. To approximate the values for these parameters, we did the following:

δ_1 = Total amount spent by Budweiser on television advertisement/ total amount spent on television commercials by all beer companies.
= 88,847,000/850,000,000 = .105

δ_2 = Total amount spent by Coors on television advertisement/total amount spent on television commercials by all beer companies.
= 57,615,000/850,000,000 = .067

The disease-free equilibrium point E_0 is the same as in the first model. It always exists and is semi-stable. E_1 has the same coordinates as the first model but this time it can be stable when $k_2 < \delta_1 k_1$ or semi-stable otherwise. E_2 follows this same pattern. It has the same coordinate as the first model and it is stable when $k_1 < \delta_2 k_2$ or semi-stable otherwise. E_3 exists under certain conditions. When $\delta_1 k_1 > k_2$ and $\delta_2 k_2 > k_1$ or when $\delta_1 k_1 < k_2$ and $\delta_2 k_2 < k_1$ the endemic point exists. Due to the complexity of evaluating the eigenvalues we used the actual values of the parameters to determine the stability. As a result, the endemic point exists and is stable (see appendix). Because E_1 and E_2 are stable under the conditions that $\delta_1 k_1 < k_2$ and $\delta_2 k_2 < k_1$, we have four equilibrium points: E_0, E_1, E_2, E_3, and E_3 is the only stable one. Therefore if we have the case when $\delta_1 k_1 > k_2$ and $\delta_2 k_2 > k_1$ then only the endemic equilibrium point is stable(see appendix).

3 Stochastic Simulation

In order to give a more realistic study of our model, we use a stochastic simulation model. Beer purchasing, to some degree, is based on random selecting by people. In our deterministic model, this is not taken into account. Intuitively, this randomness can be considered in a stochastic simulation where we use a continuous time discrete stochastic process to introduce randomness. In our model there are twelve events that can occur. We assume that the waiting time to the next event is exponentially distributed. Because of the enormous amount of time between events, we instead count the number of events that occur per day, which is a Poisson random variable, this is called discretization.

We used the five differential equations of the deterministic model to determine the five rates. These rates become the rates of exponential distributions. We choose our unit of time to be one day, and thus the number of events that happen in one day are Poisson random variables with parameters θ_i, $i = 1, 2, 3, 4, 5$, corresponding to every one of the five differential equations. The state of the process is updated and we obtain the new rates. This process creates the stochastic simulations.

4 Conclusion

Recall that for the first model the four equilibrium points that we found always exist. Depending on the initial conditions, any of the four cases mentioned earlier in the analysis can occur.

Now interpretations are in order of what the equilibrium points mean to our specific Budweiser-Coors Light situation. First, recall that when a point is semi-stable, for certain initial conditions close to the equilibrium point, the solution will tend to that point. For some other conditions, though the solution will move away from the equilibrium point.

If the initial conditions are near E_3, both Budweiser and Coors Light will prosper from the television market. That is, they will both co-exist. From the results of our graphs, we find that at the values for our parameters Budweiser controls more of the television market than Coors Light by about 3 million consumers.

The interesting thing to point out is that depending on the initial conditions, one of the two beer companies, Budweiser or Coors Light may control the television market. If this were to happen, it would be for values close to E_1 or E_2.

Since E_0 is the infection-free state, it is not necessary to study it in detail since the I_i and T_i values are all zero. E_1 and E_2 on the other hand are more engaging to study. Recall that E_1 is semi-stable. At E_1, for certain initial conditions, $I_2 = 0$ and $I_1 > 0$, and Coors Light is not selling beer (*i. e.* their commercials are not influencing consumers to buy). So, even though Coors Light is not bankrupt, we conclude that the brand has very low sales. This of course only occurs for certain initial conditions since E_1 is semi-stable. The exciting thing is that it is possible to find initial conditions that tend to this particular equilibrium point. The converse is true for E_2, (*i. e.* Budweiser does not succeed in dominating the beer market via television advertising). Therefore, it is possible for Coors Light to dominate the television market.

In the second model we observed different conclusions. Using the actual values of the parameters we conclude that Budweiser and Coors Light will continue to co-exist. Also we see that Budweiser attracts a higher number of beer drinkers than Coors Light and has a higher number of commercials. In the analysis, we predict cases in which one company will control the consumers who purchase beer due to television advertisements. If $\delta_1 k_1 > k_2$ then Coors Light will approach zero or close to zero consumers purchasing beer. If $\delta_2 k_2 > k_1$ then Budweiser will approach a state that has zero or close to zero consumers purchasing beer. This implies that the more money companies spend on television commercials the more consumers it attracts. It also shows the minimum amount of television advertising money needed for one brand to overtake the top brand or stay on top as the leading company. In order to find the minimum cost, one of the companies should estimate the carrying capacity (k_i) of the competing company (possibly through a statistic model) and estimate the maximum carrying capacity value for the next year (k_m). The minimum cost that will ensure the success of the company is measured as follows: cost_{min} = cost per commercials $\times$ $\delta_1 k_1 > k_m$. As per our model Budweiser will do better than Coors Light at a minimum cost of advertising.

Through our stochastic simulation we concluded that S, I_1, I_2, T_1, T_2, oscillated around the equilibrium point we obtained in the deterministic model, E_3. This gives a more realistic support of our deterministic results since consumer purchasing is based on random decision.

4.1 Future Research

Even though we took a specific example with Budweiser and Coors Light, this model can take any two competing products or species and construct a relationship between them. A number of variations can be added to our model to make it more realistic. For instance, we can include a quality factor for the brands of beer. The loyalty of the beer drinkers towards a particular brand can also be considered. We can also change the delay time to be different for the competing products. Looking at the future to make the model as realistic as possible, we can make all parameters be functions of time. Even though this will make the model very complicated, it can produce very interesting results.

Acknowledgments

This study was supported by the following institutions and grants: National Science Foundation (NSF Grant DMS-9600027); National Security Agency (NSA Grants MDA 904-96-1-0032 and MDA

904-97-1-0074); Presidential Faculty Fellowship Award (NSF Grant DEB 925370) and Presidential Mentoring Award (NSF Grant HRD 9724850) to Carlos Castillo-Chavez; Office of the Provost of Cornell University, MTBI, and Dirección de la División de Ciencias Básicas e Ingeniería, UAM-Iztapalapa, México.

Thanks a million to the following people who helped us along the way of getting our Bud vs. Coors Light project finished: Carlos Castillo-Chavez, Jorge Velasco, Carlos Hernández, Phil Cavalier, Mabel Torres, Julio Villarreal, Anheuser-Busch customer relations, David Sánchez, Dámaris Santana.

References

1. Berman, Oded, Krass, Dmitry and XU Chein Wei. 1997. Generalized flow-interception facility location models with probablistic customer flows. *Communications In Statistics: Stochastic Models* **12**(1): 1-25.

2. Budweiser customer relations. Interview with customer relations associate. 1997.

3. Grimmett, G.R. and D. R. Stirzaker. 1992. *Probability and Random Processes* Oxford: Clarendon Press.

4. Kao, Edward P.C. 1997. *An introduction to stochastic processes* United States: Wadson Publishing Company.

5. Mc Graw, Dan. 1995. Rocky Mountain High. *Business* **119**:67-68.

6. Melcher, Richard A. 1996. Is It Finally Miller Time? *Business Week* **3462**:37.

7. Rachev, Svetlozart T. 1991. *Probability metrics and the stability of stochastic models* England: John Wiley & Sons Ltd.

8. Recent Liquor Industry Statistics. 1997. http://www.health.org/pressrel/jan97/6.htm

9. *The U.S. beer market: impact databank review and forecast* 1994. New York: M. Shanken Communications, Inc.

10. *The U.S. beer market: impact databank review and forecast* 1995. New York: M. Shanken Communications, Inc.

11. Vilcassim, J. Naufel and Jain C. Dipak. 1994. Estimating household purchase rates for consumer nondurable goods. *Applied Stochastic Models and Data Analysis* **10**(1):15-26

12. Woodruff, David, Richard A. Melcher, Sandra Dallas. 1995. From the microbrewers who brought you Bud, Coors. *Business Week* **3421**:66-70.

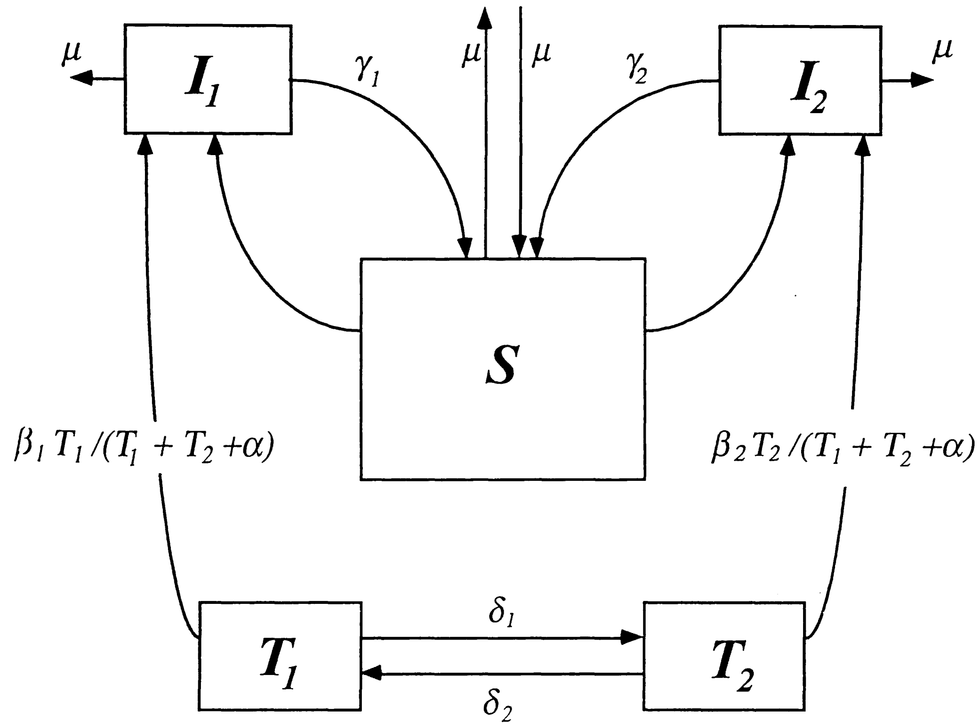

Figure 1. Graphical representation of the model

- Appendix

- MODEL #1

Finding E(0) and stability

- A Jacobian Matrix is used to linearize and approximate the system

- Setting the four differential equations to zero we obtained the following solutions

```
I1 = 0;
I2 = 0;
T1 = 0;
T2 = 0;
```

- This is the Jacobian Matrix evaluated at I1=I2=T1=T2=0

$$\begin{pmatrix} -g1-m & 0 & \frac{B1\,P}{a} & 0 \\ 0 & -g2-m & 0 & \frac{B2\,P}{a} \\ 0 & 0 & r1 & 0 \\ 0 & 0 & 0 & r2 \end{pmatrix}$$

Eigenvalues[A]

$\{-g1-m,\ -g2-m,\ r1,\ r2\}$

- These eigenvalues were obtained to determine stability. In this case we see that it is semi-stable due to the negative and postive eigenvalues.

- Finding E(1) and stability

- The following equilibrium point is used to evaluate the matrix and determine the Jacobian

$$I1 = -\frac{B1\,k1\,(I2-P)}{a\,g1+B1\,k1+g1\,k1+a\,m+k1\,m};$$

```
I2 = 0;
T1 = k1;
T2 = 0;
```

Matrix[A1]

$$\begin{pmatrix} -g1-\frac{B1\,k1}{a+k1}-m & -\frac{B1\,k1}{a+k1} & -\frac{B1\,k1\,\left(P-\frac{B1\,k1\,P}{a\,g1+B1\,k1+g1\,k1+a\,m+k1\,m}\right)}{(a+k1)^2}+\frac{B1\,\left(P-\frac{B1\,k1\,P}{a\,g1+B1\,k1+g1\,k1+a\,m+k1\,m}\right)}{a+k1} & -\frac{B1\,k1\,\left(P-\frac{B1\,k1}{a\,g1+B1\,k1+g1}\right)}{(a+k1)^2} \\ 0 & -g2-m & 0 & \frac{B2\,\left(P-\frac{B1\,k1\,P}{a\,g1+B1\,k1+g1\,k1}\right)}{a+k1} \\ 0 & 0 & -r1\left(1-\frac{B1\,k1\,P}{(a\,g1+B1\,k1+g1\,k1+a\,m+k1\,m)\left(\frac{B1\,k1\,P}{a\,g1+B1\,k1+g1\,k1+a\,m+k1\,m}+w\right)}\right) & 0 \\ 0 & 0 & 0 & r2 \end{pmatrix}$$

Eigenvalues[A1]

$$\left\{-g2-m,\ -g1-\frac{B1\,k1}{a+k1}-m,\ r2,\ -r1\left(1-\frac{B1\,k1\,P}{(a\,g1+B1\,k1+g1\,k1+a\,m+k1\,m)\left(\frac{B1\,k1\,P}{a\,g1+B1\,k1+g1\,k1+a\,m+k1\,m}+w\right)}\right)\right\}$$

- From these eigenvalues we observe that all the eigenvalues but the third are negative. Therfore, this equilibria is semi-stable.

- Finding E(2) and stability

- The following equilibrium point is used to evaluate the matrix and determine the Jacobian

$$I1 = 0;$$
$$I2 = \frac{B2\,k2\,P}{a\,g2+B2\,k2+g2\,k2+a\,m+k2\,m};$$
$$T1 = 0;$$
$$T2 = k2;$$

MatrixForm[A2]

$$\begin{pmatrix} -g1-m & 0 & \frac{B1\,\left(P-\frac{B2\,k2\,P}{a\,g2+B2\,k2+g2\,k2+a\,m+k2\,m}\right)}{a+k2} & 0 \\ -\frac{B2\,k2}{a+k2} & -g2-\frac{B2\,k2}{a+k2}-m & -\frac{B2\,k2\,\left(P-\frac{B2\,k2\,P}{a\,g2+B2\,k2+g2\,k2+a\,m+k2\,m}\right)}{(a+k2)^2} & -\frac{B2\,k2\,\left(P-\frac{B2\,k2\,P}{a\,g2+B2\,k2+g2\,k2+a\,m+k2\,m}\right)}{(a+k2)^2}+\frac{B2\,\left(P-\frac{B2\,k2}{a\,g2+B2\,k2+g2\,k2}\right)}{a+k2} \\ 0 & 0 & r1 & 0 \\ 0 & 0 & 0 & -r2\left(1-\frac{B2\,k2\,P}{(a\,g2+B2\,k2+g2\,k2+a\,m+k2\,m)\left(\frac{B2\,k2\,P}{a\,g2+B2\,k2+g2\,k2+a}\right.}\right. \end{pmatrix}$$

Eigenvalues[A2]

$$\left\{-g1-m,\ -g2-\frac{B2\,k2}{a+k2}-m,\ r1,\ -r2\left(1-\frac{B2\,k2\,P}{(a\,g2+B2\,k2+g2\,k2+a\,m+k2\,m)\left(\frac{B2\,k2\,P}{a\,g2+B2\,k2+g2\,k2+a\,m+k2\,m}+w\right)}\right)\right\}$$

- From these eigenvalues we observe that all the eigenvalues but the third are negative. Therfore, because of the third positive eigenvalue, this equilibria is semi-stable.

- Finding E(3) and stability

- The following equilibrium point is used to evaluate the matrix and determine the Jacobian

```
T1 = k1;
T2 = k2;
```

$$I1 = \frac{B1\,k1\,(g2 + m)\,P}{B1\,k1\,(g2 + m) + a\,(g1 + m)\,(g2 + m) + (g1 + m)\,(B2\,k2 + (k1 + k2)\,(g2 + m))};$$

$$I2 = \frac{B2\,k2\,(g1 + m)\,P}{B1\,k1\,(g2 + m) + a\,(g1 + m)\,(g2 + m) + (g1 + m)\,(B2\,k2 + (k1 + k2)\,(g2 + m))};$$

```
FullSimplify[MatrixForm[A3]]
```

$$\begin{pmatrix} -g1 - \frac{B1\,k1}{a+k1+k2} - m & -\frac{B1\,k1}{a+k1+k2} & \frac{B1\,(a+k2)\,(g1+m)\,(g2+m)\,P}{(a+k1+k2)\,(B1\,k1\,(g2+m)+a\,(g1+m)\,(g2+m)+(g1+m)\,(B2\,k2+(k1+k2)\,(g2+m)))} \\ -\frac{B2\,k2}{a+k1+k2} & -g2 - \frac{B2\,k2}{a+k1+k2} - m & -\frac{B2\,k2\,(g1+m)\,(g2+m)\,P}{(a+k1+k2)\,(B1\,k1\,(g2+m)+a\,(g1+m)\,(g2+m)+(g1+m)\,(B2\,k2+(k1+k2)\,(g2+m)))} \\ 0 & 0 & -\frac{r1\,((g2+m)\,(B1\,k1+(a+k1+k2)\,(g1+m))\,w+B2\,k2\,(g1+m)\,(P+w))}{B1\,k1\,(g2+m)\,(P+w)+(g1+m)\,((a+k1+k2)\,(g2+m)\,w+B2\,k2\,(P+w))} \\ 0 & 0 & 0 \end{pmatrix}$$

```
Eigenvalues[A3]
```

$$\Big\{-\frac{1}{2\,(a + k1 + k2)}\Big(B1\,k1 + B2\,k2 + (a + k1 + k2)\,(g1 + g2 + 2\,m) + \sqrt{((B1\,k1 + B2\,k2 + (a + k1 + k2)\,(g1 + g2 + 2\,m))^2 - 4\,(a + k1 + k2)\,(B1\,k1\,(g2 + m) + a\,(g1 + m)\,(g2 + m) + (g1 + m)\,(B2\,k2 + (k1 + k2)\,(g2 + m))))}\Big),$$

$$-\frac{1}{2\,(a + k1 + k2)}\Big(B1\,k1 + B2\,k2 + (a + k1 + k2)\,(g1 + g2 + 2\,m) - \sqrt{((B1\,k1 + B2\,k2 + (a + k1 + k2)\,(g1 + g2 + 2\,m))^2 - 4\,(a + k1 + k2)\,(B1\,k1\,(g2 + m) + a\,(g1 + m)\,(g2 + m) + (g1 + m)\,(B2\,k2 + (k1 + k2)\,(g2 + m))))}\Big),$$

$$-\frac{r2\,((g1 + m)\,(B2\,k2 + (a + k1 + k2)\,(g2 + m))\,w + B1\,k1\,(g2 + m)\,(P + w))}{B1\,k1\,(g2 + m)\,(P + w) + (g1 + m)\,((a + k1 + k2)\,(g2 + m)\,w + B2\,k2\,(P + w))},$$

$$-\frac{r1\,((g2 + m)\,(B1\,k1 + (a + k1 + k2)\,(g1 + m))\,w + B2\,k2\,(g1 + m)\,(P + w))}{B1\,k1\,(g2 + m)\,(P + w) + (g1 + m)\,((a + k1 + k2)\,(g2 + m)\,w + B2\,k2\,(P + w))}\Big\}$$

- E(3) is always stable because as seen above, the last two eigenvalues are always negative. The first eigenvalue is also negative given that the radical is positive. If not, its real part is always negative. The second eigenvalue also has a negative real part given that its radical is negative. If the radical is postive then the following holds:

  For the second eigenvalue:

  $$-\frac{1}{2\ (a+k1+k2)}\left(B1\ k1 + B2\ k2 + (a + k1 + k2)\ (g1 + g2 + 2\ m) - \sqrt{(}(B1\ k1 + B2\ k2 + (a + k1 + k2)\ (g1 + g2 + \right.$$
  $$4\ (a + k1 + k2)\ (B1\ k1\ (g2 + m) + a\ (g1 + m)\ (g2 + m) + (g1 + m)\ (B2\ k2 + (k1 + k2)\ (g2 + m))))$$

  let α= (a+k1+k2)

  β= B1 k1+B2 k2+(a+k1+k2) (g1+g2+2 m)

  γ= (a+k1+k2) (B1 k1 (g2+m)+a (g1+m) (g2+m)+(g1+m) (B2 k2+(k1+k2) (g2+m))

  then the second eigenvalue takes the form:

  $$\frac{-\beta+\sqrt{\beta^2-4\gamma}}{2\alpha}$$

  $\beta^2 \geq \beta^2 - 4\gamma \geq 0$ hence: $\beta \geq \sqrt{\beta^2 - 4\gamma}$

  therefore $\frac{-\beta+\sqrt{\beta^2-4\gamma}}{2\alpha} \leq 0$. This proves E(3) is always stable.