COURSE 694, LINEAR MODELS AND VARIANCE COMPONENTS: OUTLINE FOR FALL 1996

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BU-1357-M

August 1996

Abstract

This is a list of topics and page numbers from the three books being used in the course.

Key Words

Multivariate normal, quadratic forms, non-central F, estimation, hypotheses, ML, REML, prediction.

			LMFUD, Chap. 7, pages 212-240	
			212-3	Notation $\{_c\}$
	LM, Chap. 2, pages 31-74		214	I and J
			216	gi A^-, Penrose A^+
38-9	Multivariate density function		217-8	$(X'X)^-$ and Theorem
	– marginal		220-1	Eqs, sols, invariance
	– conditional		224-5	gi of partitioned
40-1	Linear transformation			
	– mean, variance, Jacobian	→	227	
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42-3	Univariate normal			
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	– marginal			
	– conditional			
47-8	Central χ^2 F and t	→	228 χ^2	
49-51	Non-central χ^2	→	229 F	
51-3	Non-central F			
54	$x'Ax = tr(x'Ax) = tr(Axx')$			
55	**Th 1** $E(x'Ax)$: and m.g.f. (42)	→	231	Th 7.2a
56	$cov(x, x'Ax)$	→	232	Th 7.2b
57	$v(x'Ax)$			Th 7.2b
	Th 2 $x'Ax \sim \chi^2$ iff	→	232	Th 7.3 Sufficiency only
58	Corollaries			
59	**Th 3** $x'Ax$ and Bx indep.	→	233	Th 7.5 Sufficiency only
	Th 4 $x'Ax$ and $x'Bx$ indep.	→	232	Th 7.4 Sufficiency only
60-4	**Th 5** Generalized Cochran			
64	Bilinear forms – form bot 64			
65	$E(x_1'A_{12}x_2)$ (49)			
	$var(x_1'A_{12}x_2)$ (52)			
66	$cov(x_1'A_{12}x_2, x_3'A_{34}x_4)$ (58)			
66-71	Singular normal			
			233-4	Hypothesis testing with F
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			236-7	Example: 1-way classification

General Linear Model

LMFUD Chapter 8, pages 241-325

	243	Model equation	(1) (2) (3)
	245	E(e) and var(e)	(7) (10)
	246-7	Normal equations and solutions	(14) (15) (16) (17)
	254-5	$E(\beta^o)$ and var (β^o)	(48) (52)
	255-6	$\hat{y}$ and $\overline{\hat{y}}$	(53) (56)
	257-8	SSE	(57) (58) (59) (60)
	258-9	E(SSE)	(62) (63)
	259-62	Partitioning S/S: SST, SSE, $R(\mu)$, $R(\beta \mid \mu)$, R^2	(64) – (72)
	262-3	Partitioning the model: model equation	(73)
	263	estimation	(77) (78)
	264	S/S	(81) (82)
☐	267	E(S/S)	(83) (84)
☐	272	extended partition	(91) (92)
	272-3	total S/S SST	(93) Table 8.4
	274	F-statistics	(99) (100)
☐	278	General hypothesis	(120) (123)
☐	280	Table 8.5	
☐	284	Estimable functions	Definitions 8.1 and 8.2
	285	BLUE	(128) – (133)
☐	287	Basic estimable functions	(139)
☐	289-90	General linear hypothesis	
	291	F-statistic	(142) – (146)
	292	Observations	
☐	294	Estimation under H	(150)
	295	Calculating Q	
	296	Anova ;	
		Non-testable - just mention	

LMFUD Chapter 8 (continued)

300 Independent and orthogonal contrasts

Balanced data (138)

Unbalanced data

$H: K'\beta = 0$

$F(H) = Q/\hat{\sigma}^2 r_X$ for $Q = (K'\beta^o)'(K'GK)^{-1}K'\beta^o$

$H: K'\beta = 0 \quad \equiv \quad H: \left\{_c \; k_i'\beta = 0\right\}_{i=1}^{s} \qquad s = r(K)$

$H_i \;\; k_i'\beta = 0$

$F(H_i) = q_i/\hat{\sigma}^2 r_X \qquad q_i = (k_i'\beta^o)^2 \Big/ k_i'Gk_i$

Result

$Q = \Sigma q_i \quad$ iff $\quad k_i'Gk_{i'} = 0 \quad \forall \quad i,i'$.

305 Restricted models: $P'\beta = \delta$: *not* a hypothesis (171)

$P'\beta$ non-estimable (176)

and below (177)

etc.

315 $OLSE(X\beta) = X(X'X)^{-}X'y$

316 $WLSE(X\beta) = X(X'WX)^{-}X'Wy$

$BLUE(X\beta) = X(X'V^{-1}X)^{-}X'V^{-1}y \quad$ for $\quad y \sim (X\beta, V) \quad$ V non-singular

GLSE

$BLUE(X\beta) \quad$ for $\quad$ V singular : see LAA paper

Class Notes — i

Sums of Squares (re-cap)

$y = X\beta + e$

$$SST = y'y = \Sigma y^2_{ijkl\cdots}$$

$$SST_m = y'y - N\bar{y}^2$$

$$SSE = (y - X\beta^o)'(y - X\beta^o) = y'y - \beta^{o\prime}X'y$$

$$R(\beta) = SST - SSE = \beta^{o\prime}X'y$$

$y = X_1\beta_1 + X_2\beta_2$

$$R(\beta_2|\beta_1) = R(\beta_2\ \beta_1) - R(\beta_1)$$

Illustration — 2-way layout

Number of Observations

Row	Column: 1 2 ⋯ j ⋯ b	Total
1, 2, ⋮, i, ⋮, a	n_{ij}	$n_{i\cdot}$
Total	$n_{\cdot j}$	$n_{\cdot\cdot}$

Data

Column: 1 2 ⋯ j ⋯ b	Total
$y_{ij\cdot}$	$y_{i\cdot\cdot}$
$y_{\cdot j\cdot}$	$y_{\cdots}$

Model $\qquad y_{ijk} = \mu + \alpha_i + \beta_j + \gamma_{ij} + e_{ijk} \qquad k = 1, 2, \cdots, n_{ij}$

Sub-model

$E(y_{ijk}) = \mu$	$R(\mu)$
$E(y_{ijk}) = \mu + \alpha_i$	$R(\mu,\alpha)$
$E(y_{ijk}) = \mu \qquad + \beta_j$	$R(\mu,\beta)$
$E(y_{ijk}) = \mu + \alpha_i + \beta_j$	$R(\mu,\alpha,\beta)$
$E(y_{ijk}) = \mu + \alpha_i + \beta_j + \gamma_{ij}$	$R(\mu,\alpha,\beta,\gamma)$

Class Notes — ii

Calculating $R(\beta) = \beta^{o'}X'y$

$$R(\mu) = N\bar{y}^2$$

$$R(\mu,\alpha) = \Sigma_i n_{i.}\bar{y}^2_{i..}$$ LM 292, LMFUD 352

$$R(\mu,\beta) = \Sigma_j n_{.j}\bar{y}^2_{.j.}$$

$$R(\mu,\alpha,\beta) = R(\mu,\alpha) + \mathbf{r}'\mathbf{C}^{-1}\mathbf{r} \quad \text{for}$$

$$\mathbf{r} = \left\{_c\ r_j\right\}_{j=1}^{b-1} \quad \text{with} \quad r_j = y_{.j.} - \Sigma_i n_{ij}\bar{y}_{i..}$$

$$\mathbf{C} = \left\{_m\ c_{jj'}\right\}_{j,j'=1}^{b-1} \quad \text{with} \quad c_{jj} = n_{.j} - \Sigma_i n^2_{ij}/n_{i.}$$

$$c_{jj'} = -\Sigma_i n_{ij}n_{ij'}/n_{i.}$$

[LM 293
LMFUD 346, 347 and 353: (67), (68),(74), (75) and (99)
102-3: (66), (67) and (72)]

$$= R(\mu,\beta) + \mathbf{u}'\mathbf{T}^{-1}\mathbf{u} \quad \text{for}$$

$$\mathbf{u} = \left\{_c\ u_i\right\}_{i=1}^{a-1} \quad \text{with} \quad u_i = y_{i..} - \Sigma_j n_{ij}\bar{y}_{.j.}$$

$$\mathbf{T} = \left\{_m\ t_{ii'}\right\}_{i,i'=1}^{a-1} \quad \text{with} \quad t_{ii} = n_{i.} - \Sigma_j n^2_{ij}/n_{.j}$$

$$t_{ii'} = -\Sigma_j n_{ij}n_{i'j}/n_{.j}$$

[LM 297 and LMFUD 353]

$$R(\mu,\alpha,\beta,\gamma) = \Sigma_i\Sigma_j n_{ij}\bar{y}^2_{ij.}$$ [LM 292 and LMFUD 100, (60)]

Class notes — iii

Analysis of Variance

Two aspects: partitioning SST

hypotheses tested

Partitioning SST = $y'y = \Sigma_i \Sigma_j \Sigma_k y^2_{ijk}$

Partition 1 [LM 294, Table 7.7a; LMFUD 111, Table 4.8I]

Term	d.f.	Sum of Squares		
μ	1	$R(\mu)$	=	$R(\mu)$
"α"	a − 1	$R(\alpha\|\mu)$	=	$R(\mu,\alpha) - R(\mu)$
"β"	b − 1	$R(\beta\|\mu,\alpha)$	=	$R(\mu,\alpha,\beta) - R(\mu,\alpha)$
"γ"	s − a − b + 1	$R(\gamma\|\mu,\alpha,\beta)$	=	$R(\mu,\alpha,\beta,\gamma) - R(\mu,\alpha,\beta)$
Residual	N − s	$SST - R(\mu,\alpha,\beta)$	=	$y'y - R(\mu,\alpha,\beta)$
Total	N	SST	=	$y'y$

Partition 2 [LM 294, Table 7.7b; LMFUD 111, Table 4.8II]

Replace lines "α" and "β" with

"$\beta\beta$"	b − 1	$R(\beta\|\mu)$	=	$R(\mu,\beta) - R(\mu)$
"$\alpha\alpha$"	a − 1	$R(\alpha\|\mu,\beta)$	=	$R(\mu,\alpha,\beta) - R(\mu,\beta)$

Hypotheses tested

Complicated

Mostly useless

e.g., $F(\alpha|\mu)$ tests $H : \alpha_i + \frac{1}{n_{i\cdot}} \Sigma_j n_{ij}(\beta_j + \gamma_{ij})$ equal $\forall$ i

[LM 307, at (100)]

$F(\gamma|\mu,\alpha,\beta)$ tests

$$H : \left\{ \begin{array}{c} \text{Any } s - a - b + 1 \text{ LIN functions of} \\ \theta_{ij,i'j'} \equiv \gamma_{ij} - \gamma_{ij'} - \gamma_{i'j} + \gamma_{i'j'} \\ \text{where such functions are either} \\ \text{estimable } \theta\text{s or estimable} \\ \text{sums or differences of } \theta\text{s} \end{array} \right\} = 0 .$$

SPECIAL CASE: all cells filled

$F(\gamma|\mu,\alpha,\beta)$ tests: $H : \textit{all } \gamma_{ij} - \gamma_{ij'} - \gamma_{i'j} + \gamma_{i'j'} = 0$.

Also SSA_w = Yates' weighted squares of means sum of squares

$$= \Sigma_i w_i(\bar{x}_{i\cdot} - \bar{x}_{[1]})^2$$

for $w_i = \left(\frac{1}{b^2}\Sigma_j \frac{1}{n_{ij}}\right)^{-1}$, $x_{ij} = \bar{y}_{ij\cdot}$

and $\bar{x}_{i\cdot} = \Sigma_j x_{ij}/b$ $\quad \bar{x}_{[1]} = \Sigma w_i \bar{x}_{i\cdot} / \Sigma w_i$

tests $H : \bar{\mu}_{i\cdot}$ all equal $\equiv$ $H : \alpha_i + \Sigma_j \gamma_{ij}/b$ all equal .

[LM 370]

Class notes — v

S A S G L M Sum of squares [LMFUD 458-465]

Type I	Type II	Type III
(Sequential)	(Adjusted)	(Σ-restricted)*
$R(\alpha\|\mu)$	$R(\alpha\|\mu,\beta)$	$(\dot{\alpha}\|\dot{\mu},\ \dot{\beta},\dot{\gamma})_\Sigma$
$R(\beta\|\mu,\alpha)$	$R(\beta\|\mu,\alpha)$	$R(\dot{\beta}\|\dot{\mu},\ \dot{\alpha},\dot{\gamma})_\Sigma$
$R(\gamma\|\mu,\alpha,\beta)$	$R(\gamma\|\mu,\alpha,\beta)$	$R(\gamma\|\mu,\ \alpha,\beta)$

* $\Sigma\alpha_i = 0$ $\quad \Sigma_j\beta_j = 0$ $\quad \Sigma_i\gamma_{ij} = 0 \quad \forall\, j$

$\Sigma_j\gamma_{ij} = 0 \quad \forall\, i$

<u>And</u> for all cells filled it is SSA_w and SSB_w .

<u>Type IV</u>

Creates "reasonable" (?) hypotheses depending on pattern of empty cells.

μ_{11}		μ_{13}	μ_{14}
μ_{21}	μ_{22}		
	μ_{32}	μ_{33}	μ_{34}

Rows: $H:\begin{cases} \mu_{33} + \mu_{34} = \mu_{13} + \mu_{14} \\ \mu_{32} = \mu_{22} \end{cases}$ [LMFUD 464]

Class notes — vi

466 Estimable functions in SAS

Example 1-way classification
three classes
$y_{ij} = \mu + \alpha_i + e_{ij} \qquad i = 1, 2, 3$.

SAS output for every sum of squares: e.g., $R(\alpha|\mu\}$

Intercept	(μ)		
A1	(α_1)	L2	(L_2)
A2	(α_2)	L3	(L_3)
A3	(α_3)	$-$L2 $-$ L3	$(-L_2 - L_3)$

Write as f: $L_2(\alpha_1) + L_3(\alpha_2) + (-L_2 - L_3)\alpha_3$

I : for any L_2 and L_3 f is estimable.
II : there are 2 Ls.
III : create 2 fs — 2 LIN sets of L_2 and L_3

e.g.,	f_1	$L_2 = 1$	$L_3 = -1$	$f_1 = \alpha_1 - \alpha_2$
	f_2	$L_2 = 1$	$L_3 = \ \ 1$	$f_2 = \alpha_1 + \alpha_2 - 2\alpha_3$

$$F\ \frac{R(\alpha|\mu)}{\hat{\sigma}^2\ 2} \quad \text{tests} \quad H: \begin{cases} f_1 = 0 \\ f_2 = 0 \end{cases} \equiv \begin{array}{l} \alpha_1 - \alpha_2 = 0 \\ \alpha_1 + \alpha_2 - 2\alpha_3 = 0 \end{array}$$

$$\equiv \begin{array}{l} \alpha_1 - \alpha_2 = 0 \\ \alpha_1 - \alpha_3 = 0 \end{array}$$

$$\equiv \alpha_1 - \alpha_2 = \alpha_3 \ .$$

True for *all* cases.

$R(\cdot|\cdot)$ r different Ls
Choose r different LIN sets of L
Create $f_1 \cdots f_r$

$$H = \begin{bmatrix} f_1 \\ \vdots \\ f_r \end{bmatrix} = 0 \ .$$

LMFUD, Chapter 10, pages 384-414

384-8 k-factor classifications, $k > 2$.

$$y_{ijklm} = \mu_{ijkl} + e_{ijklm}$$

$$\hat{\mu}_{ijkl} = \bar{y}_{ijkl\cdot}$$

Every linear combination of the μs corresponding to filled cells is estimable:

$$\text{BLUE}(\mathbf{k}'\boldsymbol{\mu}) = \mathbf{k}'\hat{\boldsymbol{\mu}} = \mathbf{k}'\bar{\mathbf{y}} \qquad \bar{\mathbf{y}} = \left\{_c\ \bar{y}_{ijkl\cdot}\right\}.$$

And can test hypotheses.

387-8 Filled cells: extend Yates weighted squares means — SAS Type IV .

388-95 A 3-way classification .

396-9 Main-effects-only models .

400-14 Models with not all interactions .

LMFUD, Chapter 11, Covariance, pages 416-

416-8 Amending an ANOVA table .

418 Eight questions. Answers:

I	p. 429, line 3 of text
II	p. 429, line 3 of text (implied)
III	p. 429, line 3 of text
IV	p. 430, line 7
V	p. 430, line 7
VI	p. 428, line 4 in paragraph after (32)
VII	For 2-way, pp. 447-454.
VIII	2+ covariates, pp. 441-7, 451-4.

General treatment.

419 $E(y) = X\beta + Zb$ Z is columns of observed covariances.

$$= \mu 1 + X_1\beta_1 + Zb$$

$$\begin{pmatrix} X'X & X'Z \\ Z'X & Z'Z \end{pmatrix}\begin{pmatrix} \tilde{\beta} \\ \hat{b} \end{pmatrix} = \begin{pmatrix} X'y \\ Z'y \end{pmatrix}.$$

421 Assume: columns of Z are LIN
columns of Z are LIN of columns of X .

422 Solutions: $M = I - X(X'X)^{-}X'$

$R = MZ$

$$\hat{b} = (R'R)^{-1}R'y \quad (17)$$

$$\tilde{\beta} = (X'X)^{-}X'(y - X\hat{b}) \quad (16)$$

Note: $R = MZ$

has column $Mz_j = [I - X(X'X)^{-}X']z_j$

$$= z_j - X(X'X)^{-}X'z_j$$

$$= z_j - \hat{z}_j \quad \text{for} \quad \hat{z}_j = X\hat{\tau} \text{ from } E(z_j) = X\tau \qquad \hat{\tau} = (X'X)^{-}X'z_j .$$

423-4	var(estimators)	(24) – (26)
424	Reductions in sum of squares .	
425	ANOVA	
426-40	1-way, 1 covariate	
441-7	1-way, 2 covariates	
447-51	2-way single slope, 1 covariate	
451-4	2-way, multiple slopes, 1 covariate	

<u>LMFUD, Chapter 6, 1-way, covariate</u>

171 Single slope model $$E(y_{ij}) = \mu_i + bz_{ij} \quad (1)$$

172 $$\hat{b} = \frac{\Sigma_i \Sigma_j (y_{ij} - \bar{y}_r)(z_{ij} - \bar{z}_{i.})}{\Sigma_i \Sigma_j (z_{ij} - \bar{z}_{i.})^2} \quad (7)$$

$$\hat{\mu}_i = \bar{y}_{i.} - \hat{b}\bar{z}_{i.}$$

$$\hat{\mu}_i + \hat{b}\bar{z}_{.} = \bar{y}_{i.} - \hat{b}(\bar{z}_{i.} - \bar{z}_{..})$$ adjusted treatment mean .

Interlude on the R( ) notation

We've used, e.g., $R(\alpha|\mu) = R(\mu,\alpha) - R(\mu)$.

Implicit in this is the fact that μ and α can (and do) exist in the same model:

$$E(y_{ij}) = \mu + \alpha_i .$$

In dealing with covariance models this is not always possible. For example, in the 1-way classification, suppose we are interested in two different models:

$$\text{single slope:} \qquad E(y_{ij}) = \mu + \alpha_i + bz_{ij} \qquad [1]$$

$$\text{multiple slope:} \qquad E(y_{ij}) = \mu + \alpha_i + b_i z_{ij} \qquad [2]$$

We might want to consider

$$R(\mu,\alpha,b_i) - R(\mu,\alpha,b) . \qquad [3]$$

Since [1] is not a sub-model of [2] we cannot write [3] as $R(b_i|\mu,\alpha,b)$ because by the convention of that notation it would be $R(\mu,\alpha,b,b_i) - R(\mu,\alpha,b)$, the first term of which does not exist. So we now define

$$\mathcal{R}(\theta|\Delta) = R(\theta) - R(\Delta) , \qquad \text{[Chap. 2, (46)}$$

for which [3] is

$$\mathcal{R}(\mu,\alpha,b_i|\mu,\alpha,b) = R(\mu,\alpha,b_i) - R(\mu,\alpha,b) .$$

And on defining $\mu_i \equiv \mu + \alpha_i$ this is

$$\mathcal{R}(\mu_i,b_i|\mu_i,b) = R(\mu_i, b_i) - R(\mu_i,b) .$$

LMFUD, Chapter 6

180 ANOVA: 1-way classification, single slope

(a) classes, then covariate (μ_i and then $\mu_i + bz_{ij}$)

(b) covariate, then classes ($\mu + bz_{ij}$, and then $\mu_i + bz_{ij}$)

S/S : as R( ) *and* calculations

Equation number in book

Hypotheses

189 Hypotheses:

	μ_i	$\mu + \alpha_i$	Sum of Squares
0	$H : \mu_i = \forall\ i$	$\alpha_i = \forall\ i$	$\mathcal{R}(\mu_i, b \mid \mu, b) = R(\alpha_i \mid \mu, b)$
*	$H : \mu_i + b\bar{z}_{i\cdot} = \forall\ i$	$\alpha_i + b\bar{z}_{i\cdot} = \forall\ i$	$\mathcal{R}(\mu_i \mid \mu) = R(\alpha_i \mid \mu)$
•	$H : \mu_i + b\mathring{z}_i = \forall\ i$	$\alpha_i + b\mathring{z}_i = \forall\ i$	

$$\left.\begin{array}{r}\text{Intra-class regression}\\ \text{Multiple slope model}\end{array}\right\} E(y_{ij}) = \mu + \alpha_i + b_i z_{ij}\,.$$

196-7 ANOVA table

208 Hypotheses

Variance Components

CHAPTER		
1.	INTRODUCTION	To be read: especially Examples 1–9, pages 7-18.
2.	HISTORY	To be read: hopefully it will be found interesting.
3.	1-WAY LAYOUT	Lectures: details, pages 44-103.
4.	BALANCED DATA	Lectures: 2-way layout, pages 118-128 Tables, pages 147-151 ML, pages 152-3
5.	ANOVA ESTIMATION FOR UNBALANCED DATA	Lectures: Basic idea, pp. 172-3 Unbiasedness, p. 173 Henderson I, pp. 182 and 189 Henderson II, pp. 191 and 201 Henderson III, pp. 203-4, 217
6.	ML and REML	Lectures: Model, pp. 232-4 ML equations, pp. 234-5, (25), (27) Sampling variances, pp. 239, (40) Computing, pp. 242 REML, pp. 249-253
7.	PREDICTION	Lectures: BP, BLP, pp. 258-266 BLUP, BU-1261-M Variances of BLUP, p. 272 Mixed model equations, pp. 275-7
8.	COMPUTING	Lectures: EM, pp. 297-303
11.	OTHER TOPICS	Lectures: MINQUE, equation (59), p. 397 MINQUE and REML