EIGENVECTOR

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Abstract

This is an invited 500-word article for an upcoming *Encyclopedia of Biostatistics*. It describes the essential features of eigenvectors.

Key Words

Eigenroot, characteristic equation, canonical form, symmetric matrix, non-symmetric matrix, diagonability theorem.

EIGENVECTOR

DEFINITION

Eigenvectors and eigenroots are features of any square matrix. For square $\mathbf{A}$, of order n, its eigenroots (see entry "eigenroot") are the n solutions for λ to what is known as the characteristic equation:

$$|\mathbf{A} - \lambda\mathbf{I}| = 0\ . \tag{1}$$

For each solution λ_i we can always find a vector, $\mathbf{u}_i$, say, such that

$$\mathbf{A}\mathbf{u}_i = \lambda_i\mathbf{u}_i, \qquad \text{or equivalently} \qquad (\mathbf{A} - \lambda_i\mathbf{I})\mathbf{u}_i = \mathbf{0}\ . \tag{2}$$

The vector $\mathbf{u}_i$ is called an eigenvector of $\mathbf{A}$ corresponding to the eigenroot λ_i. Sometimes "characteristic" (or even, old-fashionably, "latent") is used in place of "eigen".

EXAMPLE

For
$$\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$$

equation (1) simplifies to $\lambda^2 - 7\lambda + 10 = 0$, giving $\lambda_1 = 2$ and $\lambda_2 = 5$ as eigenroots. The second equation in (1) is satisfied as follows:

$$\begin{bmatrix} 3-2 & 1 \\ 2 & 4-2 \end{bmatrix}\begin{bmatrix} \text{a} \\ \text{-a} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 3-5 & 1 \\ 2 & 4-5 \end{bmatrix}\begin{bmatrix} \text{b} \\ 2\text{b} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{3}$$

Thus $\mathbf{u}_1 = [\text{a} \quad -\text{a}]'$ is the eigenvector corresponding to $\lambda_1 = 2$; and $\mathbf{u}_2 = [\text{b} \quad 2\text{b}]'$ corresponds to $\lambda_2 = 5$.

GENERAL PROPERTIES

For scalar c, an eigenvector of $\mathbf{A}$ is also an eigenvector of c$\mathbf{A}$. This is so because $\mathbf{Au} = \lambda\mathbf{u}$ implies $c\mathbf{Au} = c\lambda\mathbf{u}$, i.e., $(c\mathbf{A})\mathbf{u} = (c\lambda)\mathbf{u}$. The latter is also $\mathbf{A}(c\mathbf{u}) = \lambda(c\mathbf{u})$, showing that if $\mathbf{u}$ is an eigenvector of $\mathbf{A}$ so is c$\mathbf{u}$. This is evident in (3) where a and b can be any scalars.

There is also the simple algebra that $\mathbf{Au} = \lambda\mathbf{u}$ gives $\mathbf{A}^2\mathbf{u} = \mathbf{A}(\mathbf{Au}) = \mathbf{A}(\lambda\mathbf{u}) = \lambda\mathbf{Au} = \lambda(\lambda\mathbf{u}) = \lambda^2\mathbf{u}$. Thus $\mathbf{u}$ as an eigenvector of $\mathbf{A}$ is also an eigenvector of $\mathbf{A}^2$. This extends to $\mathbf{u}$ being an eigenvector of any integer power of $\mathbf{A}$ (and negative powers for non-singular $\mathbf{A}$).

Because every eigenroot λ_i has a corresponding eigenvector $\mathbf{u}_i$

$$\mathbf{A}[\mathbf{u}_1 \ \ \mathbf{u}_2 \ \cdots \ \mathbf{u}_i \ \cdots \ \mathbf{u}_n] = [\lambda_1\mathbf{u}_1 \ \ \lambda_2\mathbf{u}_2 \ \cdots \ \lambda_i\mathbf{y}_i \ \cdots \ \lambda_n\mathbf{u}_n] \ ; \tag{4}$$

and on defining $\mathbf{U} = [\mathbf{u}_1 \ \ \mathbf{u}_2 \ \cdots \ \mathbf{u}_i \ \cdots \ \mathbf{u}_n]$ and $\mathbf{D}$ as the diagonal matrix of the λs (4) is

$$\mathbf{AU} = \mathbf{UD}\,. \tag{5}$$

CALCULATION

For eigenroot λ_i, $\mathbf{A} - \lambda_i\mathbf{I}$ is always a singular matrix. The theory of solving linear equations then yields a solution for $\mathbf{u}_i$ to (2) as $(\mathbf{A} - \lambda_i\mathbf{I})^{-}\mathbf{z}$ for $(\mathbf{A} - \lambda_i\mathbf{I})^{-}$ being a generalized inverse [see "Matrix Algebra" entry] of $\mathbf{A} - \lambda_i\mathbf{I}$, and $\mathbf{z}$ being an arbitrary vector of order n.

MULTIPLE EIGENROOTS

Since (1) is a polynomial equation of order n it has n solutions for λ, which need not be all different. Suppose λ_t^* is a root m_t^* times, for $t = 1, \cdots, k$ for $\lambda_1^* \cdots \lambda_k^*$ being all different. Then m_t^* is called the multiplicity of λ_t^*; and $\sum_{t=1}^{k} m_t^* = n$. When $(\mathbf{A} - \lambda_t^*\mathbf{I})$ has rank $n - m_t^*$ one can always find m_t^* linearly independent eigenvectors corresponding to λ_t^*. When this rank property holds for all $t = 1, \cdots, k$ (and it always holds whenever $m_t^* = 1$), then all eigenvectors are linearly independent and $\mathbf{U}$ is nonsingular.

NON-SYMMETRIC MATRICES

For non-symmetric $\mathbf{A}$ it is the preceding rank condition (known as the diagonability theorem, e.g., Searle, 1982, page 305) which determines whether $\mathbf{U}$ is nonsingular or not. When it is nonsingular (5) yields $\mathbf{D} = \mathbf{U}^{-1}\mathbf{AU}$ and $\mathbf{D}$ is known as the canonical form under similarity, or equivalently as the similar canonical form. Likewise $\mathbf{A} = \mathbf{UDU}^{-1}$ and $\mathbf{A}^r = \mathbf{UD}^r\mathbf{U}^{-1}$.

SYMMETRIC MATRICES

For symmetric $\mathbf{A}$

(i) All λ_i and $\mathbf{u}_i$ are real.

(ii) $\mathbf{U}$ is always nonsingular.

(iii) Eigenvectors are pairwise orthogonal: $\mathbf{u}_i'\mathbf{u}_j = 0$ for $i \neq j$.

(iv) Each $\mathbf{u}_i$ can be standardized to be a unit vector $\mathbf{v}_i = \mathbf{u}_i/\sqrt{\mathbf{u}_i'\mathbf{u}_i}$ so that $\mathbf{v}_i'\mathbf{v}_i = 1$.

(v) Replacing each $\mathbf{u}_i$ in $\mathbf{U}$ by $\mathbf{v}_i$ makes $\mathbf{U}$ orthogonal: $\mathbf{U}'\mathbf{U} = \mathbf{U}\mathbf{U}' = \mathbf{I}$.

(vi) $\mathbf{D} = \mathbf{U}'\mathbf{A}\mathbf{U}$ is called the canonical form under orthogonal similarity; and $\mathbf{A} = \mathbf{U}\mathbf{D}\mathbf{U}'$, $= \sum_{i=1}^{n} \lambda_i \mathbf{u}_i \mathbf{u}_i'$, the latter being known as the spectral decomposition of $\mathbf{A}$.

These properties are important to statistics wherein symmetric matrices occur in a variety of situations; e.g., dispersion matrices, and $\mathbf{X}'\mathbf{X}$ in linear models.

REFERENCE

Searle, S.R. (1982) *Matrix Algebra Useful For Statistics.* Wiley, New York.

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