EIGENROOT

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Abstract

This is an invited 500-word article for an upcoming *Encyclopedia of Biostatistics*. It describes the general features of eigenroots.

Key Words

Square matrix, determinant, trace, principal components, discriminant analysis.

EIGENROOT

DEFINITION

Eigenroots are a feature of square matrices — not of rectangular matrices. The definition and calculation of eigenroots involves determinants. The eigenroots of square matrix **A**, of order n, are the n solutions for λ to what is called the *characteristic equation* of **A**, namely,

$$|\mathbf{A} - \lambda\mathbf{I}| = 0 \; ; \tag{1}$$

i.e., the determinant of $\mathbf{A} - \lambda\mathbf{I}$ is equated to zero.

The nature of the determinant of a matrix (see entry "Matrix Algebra") is such that for **A** of order n, (1) is a polynomial equation of order n, thus having n solutions for λ. Those solutions are the eigenroots of **A**. As an example, for $\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$

$$|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} 3-\lambda & 1 \\ 2 & 4-\lambda \end{vmatrix} = (3-\lambda)(4-\lambda) - 2 = \lambda^2 - 7\lambda + 10$$

and so the characteristic equation is

$$\lambda^2 - 7\lambda + 10 = 0 \; ,$$

to which solutions are $\lambda = 2$ and 5. Thus 2 and 5 are the eigenroots of **A**.

GENERAL PROPERTIES

1. An eigenroot, through being a solution of a polynomial equation, can be positive, negative, zero, real or complex.

2. Eigenroots of a matrix need not be all different. If λ^* is a root m^* times it is said to be a multiple root with multiplicity m^*.

3. For scalar c, an eigenroot of c**A** is c(eigenroot of **A**).

4. When λ is an eigenroot of **A**, then λ^r is an eigenroot of $\mathbf{A}^r$, for r being zero or a positive integer. This extends to a polynomial function of **A**: p(**A**) has p(λ) as an eigenroot.

5. The sum of all n eigenroots of **A** of order n equals trace of **A**, the sum of its diagonal elements.

6. The product of all n eigenroots of **A** is the determinant of **A**.

SPECIAL CASES

I. Non-singular matrices: all eigenroots are non-zero.

II. Inverse matrices: $\mathbf{A}^{-1}$ has eigenroot $1/\lambda$ where λ is an eigenroot of $\mathbf{A}$.

III. Positive (semi)definite matrices: all eigenroots are positive (zero or positive).

IV. Symmetric matrices: all eigenroots are real and the number of non-zero eigenroots equals the rank of the matrix.

V. Orthogonal matrices: eigenroots come in pairs λ and $1/\lambda$, with one root being ± 1 when the matrix is of odd order.

VI. Idempotent matrices: all eigenroots are $+1$ or zero; the number that are $+1$ is the rank of the matrix.

Two uses of eigenroots in multivariate statistics are of particular importance. One is in principal components analysis applied to a vector $\mathbf{X}$ of random variables. The component (linear combination) of these variables having the r'th largest variance is then $\boldsymbol{\beta}_r'\mathbf{X}$, its variance being λ_r, the r'th largest eigenroot of the variance-covariance matrix of $\mathbf{X}$; and $\boldsymbol{\beta}_r$ is the eigenvector (see entry "eigenvector") corresponding to λ_r. In this way one can rank the principal components according to the size of their variances.

Another use of eigenroots is in discriminant analysis where one wants to classify observational units on the basis of a vector of variables, $\mathbf{X}$, say, measured on each unit. This is done using some $\mathbf{t}'\mathbf{X}$, often through maximizing the ratio of two quadratic forms, $\mathbf{t}'\mathbf{A}\mathbf{t}$ and $\mathbf{t}'\mathbf{B}\mathbf{t}$, say. This is achieved by choosing λ and $\mathbf{t}$ so that $(\mathbf{B} - \lambda\mathbf{A})\mathbf{t} = \mathbf{0}$, or equivalently $(\mathbf{A}^{-1}\mathbf{B} - \lambda\mathbf{I})\mathbf{t} = 0$. Thus λ is the eigenroot of $\mathbf{A}^{-1}\mathbf{B}$ and $\mathbf{t}$ a corresponding eigenvector.

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