3-D GEOMETRY: A TRIANGLE-ORIENTED PROOF OF THE COSINE OF THE ANGLE BETWEEN TWO VECTORS

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ABSTRACT

The well-established formula for the cosine of the angle between two vectors in 3-dimensional space is established using straightforward results of triangle geometry.

KEY WORDS

Cartesian coordinates, right-angle triangles, rotation of axes.
INTRODUCTION

Preparation of a brief article for an encyclopedia on the cosine of the angle between two vectors led first to locating the results as a definition. For vectors \( r \) and \( s \) in n-space, in which 
\[
r = [r_1 \ r_2 \ \cdots \ r_i \ \cdots \ r_n]' \quad \text{and} \quad s = [s_1 \ s_2 \ \cdots \ s_i \ \cdots \ s_n]'
\] 
are points, respectively, we find in Johnson and Wichern (1988, p. 69) that the angle \( \theta \) between \( r \) and \( s \) is defined by
\[
\cos \theta = \frac{r's}{\sqrt{r'r(s's)}}.
\] (1)

Geometrically this may not seem very satisfactory because although two vectors in 2-space always intersect (unless they are parallel), two vectors in 3-space may not; and (1) seems devoid of any requirement that vectors \( r \) and \( s \) intersect (except, of course, that even in n-space it is often assumed that all vectors start at the origin of the coordinates). Thus, as a definition of an angle, (1) may be satisfactory although not necessarily of an angle in the usual 2-dimensional sense. This prompts consideration of (1) for 2-space and 3-space. The former is easy, whereas the latter is a little more difficult; it is given in McCrea (1947, p. 9) in terms of projections, but it can also be developed in terms of triangle geometry, which is the purpose of this note.

2-SPACE

Figure 1 shows coordinates \( x_1, y_1 \) and \( x_2, y_2 \) of points of two vectors in 2-space.

\[
\begin{align*}
\cos \theta &= \cos \alpha_1 \cos \alpha_2 + \sin \alpha_1 \sin \alpha_2 \\
&= \frac{x_1x_2}{d_1d_2} + \frac{y_1y_2}{d_1d_2} = \frac{x_1x_2 + y_1y_2}{d_1d_2},
\end{align*}
\] (2)

where \( d_i = \sqrt{x_i^2 + y_i^2} \) for \( i = 1, 2 \). And (2) is the 2-dimensional form of (1).
3-SPACE

The diagram for 3-space is Figure 2 (see page 4). It is comparable to Figure 1 but a little more complicated. 0 is the origin of the Cartesian coordinates x, y and z. P₁ and P₂ with coordinates \((x₁,y₁,z₁)\) and \((x₂,y₂,z₂)\), respectively, are two points, with 0P₁ and 0P₂ being the vectors, intersecting at 0, and thus defining a 2-dimensional plane. \(\theta\), the angle of interest, is the angle between the vectors in that plane. We show that

\[
\cos \theta = \frac{x₁x₂ + y₁y₂ + z₁z₂}{d₁d₂}
\]

with

\[
dₐ = \sqrt{xₐ² + yₐ² + zₐ²} \text{ for } i = 1 \text{ and } 2.
\]

In Figure 2, perpendiculars to the x,y plane through P₁ and P₂ meet the plane at R₁ and R₂, respectively. A plane through P₂ parallel to the x,y plane is intersected by P₁R₁ at S. SQ₂P₂Q₁ is a rectangle in that plane.

We derive \(\cos \theta\) by finding the length of each side of the triangle P₁0P₂. First, from the rectangle having diagonal 0R₁ its length \(|0R₁|\) is given by

\[
|0R₁|² = x₁² + y₁².
\]

Therefore from the triangle OR₁P₁

\[
|0P₁|² = (x₁² + y₁²) + z₁² = x₁² + y₁² + z₁² = d₁².
\]

Similarly

\[
|0P₂|² = x₂² + y₂² + z₂² = d₂².
\]

Consideration of the rectangle SQ₂P₂Q₁ reveals that

\[
|SQ₁| = x₂ - x₁, \quad |SQ₂| = y₂ - y₁
\]

and clearly

\[
|SP₁| = z₂ - z₁.
\]

Therefore

\[
|SP₂|² = (x₂ - x₁)² + (y₂ - y₁)²
\]

and so

\[
|P₁P₂|² = (x₂ - x₁)² + (y₂ - y₁)² + (z₂ - z₁)²
\]

\[
= w², \text{ say}.
\]

We now have \(\Delta P₁0P₂\) having sides \(d₁,d₂\) and \(w\), in which we drop a perpendicular from P₁ on to
Figure 2. Two vectors $0P_1$ and $0P_2$ in 3-space.
Then
\[ \cos \theta = \frac{t}{d_1} \]
and
\[ |P_1T|^2 = d_1^2 - t^2 = w^2 - (d_2 - t^2) \]
which yields
\[ t = \frac{(d_1^2 + d_2^2 - w^2)}{2d_2} \].
Substituting for \( d_1^2 \) and \( d_2^2 \) from (3) and for \( w^2 \) from (4) gives
\[ t = \frac{x_1x_2 + y_1y_2 + z_1z_2}{d_2} \]
and so
\[ \cos \theta = \frac{x_1x_2 + y_1y_2 + z_1z_2}{d_1d_2} \]
in keeping with (1) for which \( r' = [x_1 \ x_2 \ x_3] \) and \( s' = [x_2 \ y_2 \ z_2] \).

**ROTATION OF AXES**

Rotating axes does not affect the angle between two vectors so its cosine is unaffected. But co-ordinates of points on the vectors get changed; and yet the formula for the cosine is the same in terms of the new co-ordinates as it is for the old. Illustration of this for the 2-space case is given in terms of Figures 4a and 4b.

In both Figures 4a and 4b (see page ) the cartesian axes are labelled \( X_1 \) and \( X_2 \), and are shown with single arrowheads. After counter-clockwise rotation through an angle \( \beta \) they are labelled \( X'_1 \) and \( X'_2 \) with two arrowheads. In both figures \( OT \) represents a vector with \( T \) having co-ordinates \( (x_1, x_2) \)
Rotation of axes through an angle $\beta$.

Figure 4a: $\beta < \text{angle TOX}_1$

Figure 4b: $\beta > \text{angle TOX}_1$
and after rotation of axes co-ordinates \((x'_1, x'_2)\). The vector is marked with three arrowheads.

In Figure 4a, the angle \(\beta\) is less than the angle between the vector and the \(X_1\) axis; in Figure 4b it is greater. In both figures TP is perpendicular to the \(X_1\) axis and TP' is perpendicular to the \(X'_1\) axis.

Hence

\[
x_1 = \text{OP}, \quad x_2 = \text{TP}, \quad x'_1 = \text{OP'}, \quad \text{and} \quad |x'_2| = \text{TP'},
\]

the latter arising from the fact that

\[
x'_2 = \text{TP'} \quad \text{in Figure 4a}, \quad \text{but} \quad x'_2 = -\text{TP'} \quad \text{in Figure 4b},
\]

where, for example, OP means the distance from O to P.

In contrast, the \(x_s\) and \(x'_s\), being co-ordinates of a point, have length and sign, \(x'_2\) being negative in Figure 4b.

We now express \(x'_1\) and \(x'_2\) in terms of \(x_1\) and \(x_2\), doing so for each figure separately. In Figure 4a

\[
x'_1 = \text{OP'} = \text{OQ'} + \text{Q'P'} = \frac{\text{OP}}{\cos \beta} + \text{TP'} \tan \beta = x'_1 \cos \beta + x'_2 \tan \beta
\]

and

\[
x'_2 = \text{TP'} = \text{TQ} - \text{P'Q} = \frac{\text{TP}}{\cos \beta} - \text{OP'} \tan \beta = x'_2 \cos \beta - x'_1 \tan \beta
\]

Therefore

\[
x'_1 = \frac{x_1 + (x_2 - x'_1 \sin \beta) \sin \beta / \cos \beta}{\cos \beta}
\]

which yields

\[
x'_1 = x_1 \cos \beta + x_2 \sin \beta
\]

and so

\[
x'_2 = \frac{x_2 - (x_1 \cos \beta + x_2 \sin \beta) \sin \beta}{\cos \beta}
\]

giving

\[
x'_2 = x_2 \cos \beta - x'_1 \sin \beta.
\]

Clearly

\[
d_{x'} = x'^2 + x'^2_2 = (x'^2_1 + x'^2_2)(\cos^2 \beta + \sin^2 \beta) + 2x_1x_2(0) = x'^2_1 + x'^2_2 = d_x
\]
as it should.

In Figure 4b, initially using \( x_2' \) as just a distance,

\[
x_1' = OP' = TR' = TR + RR' = \frac{TP}{\sin \beta} + OP' \cot \beta = \frac{x_2}{\sin \beta} + x_2' \cot \beta
\]

\[
= \frac{x_2 + x_2' \cos \beta}{\sin \beta}
\]

and

\[
x_2' = TP' = OR' = OS - R'S = \frac{OP}{\sin \beta} - \frac{TR'}{\tan \beta} = \frac{x_1'}{\sin \beta} - \frac{x_1'}{\tan \beta}
\]

\[
= \frac{x_1 - x_1' \cos \beta}{\sin \beta}.
\]

Hence

\[
x_1' = \frac{x_2 + (x_1 - x_1' \cos \beta) \cos \beta / \sin \beta}{\sin \beta}
\]

giving

\[
x_1' = x_1 \cos \beta + x_2 \sin \beta.
\]

Thus

\[
x_2' = \frac{x_1 - (x_1 \cos \beta + x_2 \sin \beta) \cos \beta}{\sin \beta}
\]

\[
= x_1 \sin \beta - x_2 \cos \beta.
\]

Now because \( x_2' \) as a co-ordinate is, in this case, the negative of its length,

\[
x_2' = x_2 \cos \beta - x_1 \sin \beta.
\]

Therefore, in both Figures 4a and 4b

\[
x_1' = x_1 \cos \beta + x_2 \sin \beta \quad \text{and} \quad x_2' = x_2 \cos \beta - x_1 \sin \beta.
\]

For a second vector, as in Figure 2, \((y_1, y_2)\) and \((y_1', y_2')\) are connected in exactly the same way as are the \( x \)s in (7). Therefore, using \( s \equiv \sin \beta \) and \( c \equiv \cos \beta \),

\[
x_1' y_1' + x_2' y_2' = (x_1 c + x_2 s)(y_1 c + y_2 s) + (x_2 c - x_1 s)(y_2 c - y_1 s)
\]

\[
= (x_1 y_1 + x_2 y_2)(c^2 + s^2) + (x_1 y_2 + x_2 y_1)(cs - cs)
\]

\[
= x_1 y_1 + x_2 y_2.
\]

Thus, on also using (6)

\[
\cos B = \frac{x_1 y_1 + x_2 y_2}{d_x d_y} = \frac{x_1' y_1' + x_2' y_2'}{d_x' d_{y'}}
\]

so illustrating that \( \cos B \) is the same function of co-ordinates as it is before rotation.
References
