Design and Analyses for a 2x4x4 Nitrogen, Potash, and Phosphorous Fertilizer Trial on Spinach

BU-132-M Walter T. Federer * April, 1961

An area of land 8 plots x 8 plots = 64 plots was available for a fertilizer trial on spinach. Since it was quite possible that gradients existed in two directions, some sort of a latin square arrangement or some sort of covariates appeared desirable. The former was decided on.

There are two levels of nitrogen denoted as zero and one, four levels of potash denoted as 00, 01, 10, and 11, and four levels of phosphorous denoted as 00, 01, 10, and 11. In this form the $2(4^2)$ factorial can be equated to a 2^5 factorial and the 8x8 latin square given in Table 32 of Yates' "The design and analysis of factorial experiments," (or a similar one) can be used. Likewise, if two factors b and c, say, are used at two levels, 0 and 1, to describe the four levels of potash, then the BC interaction is the quadratic effect (see Snedecor, G. W., "Statistical Methods, 4th ed., page 410). (It is possible that the quadratic effects of potash and phosphorous are important in this experiment, but it was anticipated that interactions with the quadratic effect would not be too important.) Also, the two factors d and e each at the zero and one levels are used to describe the 4 phosphorous levels.

The effects ABC, ADE, and BCDE are confounded with rows 1 to 4 of an 8x8 latin square; the effects ABD, BCE, and ACDE with rows 5 to 8; the effects ACE, BCD, and ABDE with columns 1 to 4; and the effects ACD, BDE, and ABCE are confounded with columns 5 to 8. The remaining effects are unconfounded with rows or columns. The schematic lay-out is presented in Table 1 using the treatment described above (i.e., the a factor = nitrogen, the b and c factors = potash, and the d and e factors = phosphorous).

The randomization procedure is to completely randomize the rows and then to completely randomize the columns. The analysis of variance is described in Table 2. The levels of the effects are obtained as described by Federer, "Experimental Design," chapter VII. It should be noted that the levels of the effects confounded with rows or columns are obtained from the rows or columns in which the effects are unconfounded (see Yates, loc. cited., page 35).

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Table 1. Schematic lay-out for an 8x8 quasi-latin square for a 2x4x4 factorial.

Levels of effects	Rows	Columns and levels of effects							
		$(ACE)_0$ $(BCD)_0$ $(ABDE)_0$	$(ACE)_0$ $(BCD)_1$ $(ABDE)_1$	$(ACE)_1$ $(BCD)_0$ $(ABDE)_1$	$(ACE)_1$ $(BCD)_1$ $(ABDE)_0$	$(ACD)_0$ $(BDE)_0$ $(ABCE)_0$	$(ACD)_1$ $(BDE)_1$ $(ABCE)_0$	$(ACD)_0$ $(BDE)_1$ $(ABCE)_1$	$(ACD)_1$ $(BDE)_0$ $(ABCE)_1$
		1	2	3	4	5	6	7	8
$(ABC)_0$, $(ADE)_0$, $(BCDE)_0$	1	00000	11001	01100	10101	11010	10110	01111	00011
$(ABC)_0$, $(ADE)_1$, $(BCDE)_1$	2	01101	10100	00001	11000	01110	00010	11011	10111
$(ABC)_1$, $(ADE)_0$, $(BCDE)_1$	3	00111	11110	01011	10010	11101	10001	01000	00100
$(ABC)_1$, $(ADE)_1$, $(BCDE)_0$	4	01010	10011	00110	11111	01001	00101	11100	10000
$(ABD)_0$, $(BCE)_0$, $(ACDE)_0$	5	11100	00101	10111	01110	00000	01011	10010	11001
$(ABD)_0$, $(BCE)_1$, $(ACDE)_1$	6	10110	01111	11101	00100	10011	11000	00001	01010
$(ABD)_1$, $(BCE)_0$, $(ACDE)_1$	7	11011	00010	10000	01001	00111	01100	10101	11110
$(ABD)_1$, $(BCE)_1$, $(ACDE)_0$	8	10001	01000	11010	00011	10100	11111	00110	01101

Table 2. Breakdown of the degrees of freedom for a 2x4x4 factorial in an 8x8 latin square.

Source of variation	d.f.		Sum of squares
Total (uncorrected)	64		$\underset{fghijkm}{\Sigma\Sigma\Sigma\Sigma\Sigma\Sigma\Sigma} Y^2_{fghijkm}$
Correction for mean	1		$Y^2_{.......}/64=CT$
Rows	7		$\sum_{f=1}^{8} \frac{Y^2_{f......}}{8} -CT$
Rows 1-4 vs 5-8	1		$\frac{(\text{Sum 1st 4 - Sum of last 4})^2}{64}$
Rows 1-4	3		SS among rows 1-4
ABC from rows 1-4		1	$[(ABC)_1-(ABC)_0]^2/32$
ADE from rows 1-4		1	$[(ADE)_1-(ADE)_0]^2/32$
BCDE from rows 1-4		1	$[(BCDE)_0-(BCDE)_1]^2/32$
Rows 5-8	3		SS among rows 5-8
ABD from rows 5-8		1	$[(ABD)_1-(ABD)_0]^2/32$
BCE from rows 5-8		1	$[(BCE)_1-(BCE)_0]^2/32$
ACDE from rows 5-8		1	$[(ACDE)_0-(ACDE)_1]^2/32$
Columns	7		$\sum_{g=1}^{8} Y^2_{.g.....}/8-CT$
Columns 1-4 vs 5-8	1		$(\text{Sum of 1st 4 - Sum of last 4})^2/64$
Columns 1-4	3		SS among 1st 4 columns
ACE from cols. 1-4		1	$[(ACE)_1-(ACE)_0]^2/32$
BCD from cols. 1-4		1	$[(BCD)_1-(BCD)_0]^2/32$
ABDE from cols. 1-4		1	$[(ABDE)_0-(ABDE)_1]^2/32$
Columns 5-8	3		SS among last 4 columns
ACD from col. 5-8		1	$[(ACD)_1-(ACD)_0]^2/32$
BDE from col. 5-8		1	$[(BDE)_1-(BDE)_0]^2/32$
ABCE from col. 5-8		1	$[(ABCE)_0-(ABCE)_1]^2/32$
Treatments (elim. rows and cols.)	31		addition of following
A (nitrogen)	1		$[Y_{..1....}-Y_{..0....}]^2/64$
B (potash)	1		$[Y_{...1...}-Y_{...0...}]^2/64$
C (potash)	1		$[Y_{....1..}-Y_{....0..}]^2/64$
BC (potash)	1		$[Y_{...11..}+Y_{...00..}-Y_{...10..}-Y_{...01..}]^2/64$

Table 2. (Cont'd)

D	phosphorous	1	$[Y_{.....1.}-Y_{.....0.}]^2/64$
E		1	$[Y_{......1}-Y_{......0}]^2/64$
DE		1	$[Y_{.....11}+Y_{.....00}-Y_{.....10}-Y_{.....01}]^2/64$
AB	A x potash	1	$[(AB)_0-(AB)_1]^2/64$
AC		1	$[(AC)_0-(AC)_1]^2/64$
ABC'		1'	$[(ABC)_1-(ABC)_0$ from rows 5-8$]^2/32$
AD	A x phosphorous	1	$[(AD)_0-(AD)_1]^2/64$
AE		1	$[(AE)_0-(AE)_1]^2/64$
ADE'		1'	$[(ADE)_1-(ADE)_0$ from rows 5-8$]^2/32$
BD	potash x phosphorous	1	$[(BD)_0-(BD)_1]^2/64$
BE		1	$[(BE)_0-(BE)_1]^2/64$
BDE'		1'	$[(BDE)_1-(BDE)_0$ from cols. 1-4$]^2/32$
CD		1	$[(CD)_0-(CD)_1]^2/64$
CE		1	$[(CE)_0-(CE)_1]^2/64$
CDE		1	$[(CDE)_1-(CDE)_0]^2/64$
BCD'		1'	$[(BCD)_1-(BCD)_0$ from cols. 5-8$]^2/32$
BCE'		1'	$[(BCE)_1-(BCE)_0$ from rows 1-4$]^2/32$
BCDE'		1'	$[(BCDE)_0-(BCDE)_1$ from rows 5-8$]^2/32$
ABD'	nitrogen x potash x phosphorous	1'	$[(ABD)_1-(ABD)_0$ from rows 1-4$]^2/32$
ABE		1	$[(ABE)_1-(ABE)_0]^2/64$
ABDE'		1'	$[(ABDE)_0-(ABDE)_1$ from cols. 5-8$]^2/32$
ACD'		1'	$[(ACD)_1-(ACD)_0$ from cols. 1-4$]^2/32$
ACE'		1'	$[(ACE)_1-(ACE)_0$ from cols. 5-8$]^2/32$
ACDE'		1'	$[(ACDE)_0-(ACDE)_1$ from rows 1-4$]^2/32$
ABCD		1	$[(ABCD)_0-(ABCD)_1]^2/64$
ABCE'		1'	$[(ABCE)_0-(ABCE)_1$ from cols. 1-4$]^2/32$
ABCDE		1	$[(ABCDE)_1-(ABCDE)_0]^2/64$
Error		18	by subtraction

For pedagogical purposes, a linear model is described for the above design and the least squares estimates of effects are obtained. The usual linear equation for yield in a trifactorial experiment with two sources of stratification is

$$Y_{fghsr}=\mu+\rho_f+\lambda_g+\alpha_h+\pi_s+\tau_r+\alpha\pi_{hs}+\alpha\tau_{hr}+\pi\tau_{sr}+\alpha\pi\tau_{hsr}+\epsilon_{fghsr}$$

This model may be rewritten as follows for a 2^5 factorial:

$$\begin{aligned}Y_{fghijkm}=\mu+\rho_f+\lambda_g&+(-1)^{h-1}\alpha+(-1)^{i-1}\beta+(-1)^{h+i}\alpha\beta+(-1)^{j-1}\gamma+(-1)^{h+j}\alpha\gamma+(-1)^{i+j}\beta\gamma\\
&+(-1)^{h+i+j-1}\alpha\beta\gamma+(-1)^{k-1}\delta+(-1)^{h+k}\alpha\delta+(-1)^{i+k}\beta\delta+(-1)^{h+i+k-1}\alpha\beta\delta\\
&+(-1)^{j+k}\gamma\delta+(-1)^{h+j+k-1}\alpha\gamma\delta+(-1)^{i+j+k-1}\beta\gamma\delta+(-1)^{h+i+j+k}\alpha\beta\gamma\delta\\
&+(-1)^{m-1}\upsilon+(-1)^{h+m}\alpha\upsilon+(-1)^{i+m}\beta\upsilon+(-1)^{h+i+m-1}\alpha\beta\upsilon+(-1)^{j+m}\gamma\upsilon\\
&+(-1)^{h+j+m-1}\alpha\gamma\upsilon+(-1)^{i+j+m-1}\beta\gamma\upsilon+(-1)^{h+i+j+m}\alpha\beta\gamma\upsilon+(-1)^{k+m}\delta\upsilon\\
&+(-1)^{h+k+m-1}\alpha\;\upsilon+(-1)^{i+k+m-1}\beta\delta\upsilon+(-1)^{h+i+k+m}\alpha\beta\delta\upsilon+(-1)^{j+k+m-1}\gamma\delta\upsilon\\
&+(-1)^{h+j+k+m}\alpha\gamma\delta\upsilon+(-1)^{i+j+k+m}\beta\gamma\delta\upsilon+(-1)^{h+i+j+k+m-1}\alpha\beta\gamma\delta\upsilon+\epsilon_{fghijkm}\,.\end{aligned}$$

In the above form, 31 restrictions have been placed in the linear equation for yield on the 31 effects from the 2^5 factorial. Thus, when the normal equations are obtained (as below), additional restrictions need only to be placed on the ρ_f and on the λ_g, e.g.,

$$\sum_{f=1}^{8}\hat{\rho}_f=0=\sum_{g=1}^{8}\hat{\lambda}_g\quad .$$

Also, in the above equation the 31 effects - $\alpha,\beta,\alpha\beta,\cdots\cdots,\alpha\beta\gamma\delta\upsilon$ - are one-half of the effects in the 2^5 factorial as described in Table 1 and in the literature cited above. The μ is an overall effect, the ρ_f are the row effects, and the λ_g are the column effects. The $\epsilon_{fghijkm}$ are independent and identically distributed random variables with mean zero and variance σ^2_ϵ. The assumption of normality of distribution is required for tests of hypotheses and construction of confidence intervals. The remaining effects are considered to be fixed effects.

The sum of squares to be minimized with respect to the 31+8+8+1 parameters is

$$\sum_{f=1}^{8}\sum_{g=1}^{8}\sum_{h=0}^{1}\sum_{i=0}^{1}\sum_{j=0}^{1}\sum_{k=0}^{1}\sum_{m=0}^{1}\epsilon^2_{fghijkm}\quad .$$

The resulting normal equations are (after imposing the restrictions $\Sigma\hat{\rho}_f=0=\Sigma\hat{\lambda}_g$):

$$\mu: \quad 64\hat{\mu} = Y_{\cdot\cdot\cdot\cdot\cdot\cdot\cdot}$$

$$\rho_1: \quad 8(\hat{\mu}+\hat{\rho}_1)-8\widehat{\alpha\beta\gamma}-8\widehat{\alpha\delta\upsilon}+8\widehat{\beta\gamma\delta\upsilon} = Y_{1\cdot\cdot\cdot\cdot\cdot\cdot}$$

$$\rho_2: \quad 8(\hat{\mu}+\hat{\rho}_2)-8\widehat{\alpha\beta\gamma}+8\widehat{\alpha\delta\upsilon}-8\widehat{\beta\gamma\delta\upsilon} = Y_{2\cdot\cdot\cdot\cdot\cdot\cdot}$$

$$\rho_3: \quad 8(\hat{\mu}+\hat{\rho}_3+\widehat{\alpha\beta\gamma}-\widehat{\alpha\delta\upsilon}-\widehat{\beta\gamma\delta\upsilon}) = Y_{3\cdot\cdot\cdot\cdot\cdot\cdot}$$

$$\rho_4: \quad 8(\hat{\mu}+\hat{\rho}_4+\widehat{\alpha\beta\gamma}+\widehat{\alpha\delta\upsilon}+\widehat{\beta\gamma\delta\upsilon}) = Y_{4\cdot\cdot\cdot\cdot\cdot\cdot}$$

$$\rho_5: \quad 8(\hat{\mu}+\hat{\rho}_5-\widehat{\alpha\beta\delta}-\widehat{\beta\gamma\upsilon}+\widehat{\alpha\gamma\delta\upsilon}) = Y_{5\cdot\cdot\cdot\cdot\cdot\cdot}$$

$$\rho_6: \quad 8(\hat{\mu}+\hat{\rho}_6-\widehat{\alpha\beta\delta}+\widehat{\beta\gamma\upsilon}-\widehat{\alpha\gamma\delta\upsilon}) = Y_{6\cdot\cdot\cdot\cdot\cdot\cdot}$$

$$\rho_7: \quad 8(\hat{\mu}+\hat{\rho}_7+\widehat{\alpha\beta\delta}-\widehat{\beta\gamma\upsilon}-\widehat{\alpha\gamma\delta\upsilon}) = Y_{7\cdot\cdot\cdot\cdot\cdot\cdot}$$

$$\rho_8: \quad 8(\hat{\mu}+\hat{\rho}_8+\widehat{\alpha\beta\delta}+\widehat{\beta\gamma\upsilon}+\widehat{\alpha\gamma\delta\upsilon}) = Y_{8\cdot\cdot\cdot\cdot\cdot\cdot}$$

$$\lambda_1: \quad 8(\hat{\mu}+\hat{\lambda}_1-\widehat{\alpha\gamma\upsilon}-\widehat{\beta\gamma\delta}+\widehat{\alpha\beta\delta\upsilon}) = Y_{\cdot 1\cdot\cdot\cdot\cdot\cdot}$$

$$\lambda_2: \quad 8(\hat{\mu}+\hat{\lambda}_2-\widehat{\alpha\gamma\upsilon}+\widehat{\beta\gamma\delta}-\widehat{\alpha\beta\delta\upsilon}) = Y_{\cdot 2\cdot\cdot\cdot\cdot\cdot}$$

$$\lambda_3: \quad 8(\hat{\mu}+\hat{\lambda}_3+\widehat{\alpha\gamma\upsilon}-\widehat{\beta\gamma\delta}-\widehat{\alpha\beta\delta\upsilon}) = Y_{\cdot 3\cdot\cdot\cdot\cdot\cdot}$$

$$\lambda_4: \quad 8(\hat{\mu}+\hat{\lambda}_4+\widehat{\alpha\gamma\upsilon}+\widehat{\beta\gamma\delta}+\widehat{\alpha\beta\delta\upsilon}) = Y_{\cdot 4\cdot\cdot\cdot\cdot\cdot}$$

$$\lambda_5: \quad 8(\hat{\mu}+\hat{\lambda}_5-\widehat{\alpha\gamma\delta}-\widehat{\beta\delta\upsilon}+\widehat{\alpha\beta\gamma\upsilon}) = Y_{\cdot 5\cdot\cdot\cdot\cdot\cdot}$$

$$\lambda_6: \quad 8(\hat{\mu}+\hat{\lambda}_6+\widehat{\alpha\gamma\delta}+\widehat{\beta\delta\upsilon}+\widehat{\alpha\beta\gamma\upsilon}) = Y_{\cdot 6\cdot\cdot\cdot\cdot\cdot}$$

$$\lambda_7: \quad 8(\hat{\mu}+\hat{\lambda}_7-\widehat{\alpha\gamma\delta}+\widehat{\beta\delta\upsilon}-\widehat{\alpha\beta\gamma\upsilon}) = Y_{\cdot 7\cdot\cdot\cdot\cdot\cdot}$$

$$\lambda_8: \quad 8(\hat{\mu}+\hat{\lambda}_8+\widehat{\alpha\gamma\delta}-\widehat{\beta\delta\upsilon}-\widehat{\alpha\beta\gamma\upsilon}) = Y_{\cdot 8\cdot\cdot\cdot\cdot\cdot}$$

$$\alpha: \quad 64\hat{\alpha} = Y_{\cdot\cdot 1\cdot\cdot\cdot\cdot} - Y_{\cdot\cdot 0\cdot\cdot\cdot\cdot} = (A)_1 - (A)_0$$

$$\beta: \quad 64\hat{\beta} = Y_{\cdot\cdot\cdot 1\cdot\cdot\cdot} - Y_{\cdot\cdot\cdot 0\cdot\cdot\cdot} = (B)_1 - (B)_0$$

$$\alpha\beta: \quad 64\widehat{\alpha\beta} = Y_{\cdot\cdot 11\cdot\cdot\cdot} + Y_{\cdot\cdot 00\cdot\cdot\cdot} - Y_{\cdot\cdot 10\cdot\cdot\cdot} - Y_{\cdot\cdot 01\cdot\cdot\cdot} = (AB)_0 - (AB)_1$$

$$\gamma: \quad 64\hat{\gamma} = Y_{\cdot\cdot\cdot\cdot 1\cdot\cdot} - Y_{\cdot\cdot\cdot\cdot 0\cdot\cdot} = (C)_1 - (C)_0$$

$$\alpha\gamma: \quad 64\widehat{\alpha\gamma} = Y_{\cdot\cdot 1\cdot 1\cdot\cdot} + Y_{\cdot\cdot 0\cdot 0\cdot\cdot} - Y_{\cdot\cdot 1\cdot 0\cdot\cdot} - Y_{\cdot\cdot 0\cdot 1\cdot\cdot} = (AC)_0 - (AC)_1$$

$$\beta\gamma: \quad 64\widehat{\beta\gamma} = Y_{\cdot\cdot\cdot 11\cdot\cdot} + Y_{\cdot\cdot\cdot 00\cdot\cdot} - Y_{\cdot\cdot\cdot 01\cdot\cdot} - Y_{\cdot\cdot\cdot 10\cdot\cdot} = (BC)_0 - (BC)_1$$

$$\alpha\beta\gamma: \quad 64\widehat{\alpha\beta\gamma}+8(\rho_3+\rho_4-\rho_1-\rho_2) = Y_{\cdot\cdot 100\cdot\cdot} + Y_{\cdot\cdot 010\cdot\cdot} + Y_{\cdot\cdot 001\cdot\cdot} + Y_{\cdot\cdot 111\cdot\cdot} - Y_{\cdot\cdot 000\cdot\cdot} - Y_{\cdot\cdot 011\cdot\cdot}$$

$$- Y_{\cdot\cdot 101\cdot\cdot} - Y_{\cdot\cdot 110\cdot\cdot} = (ABC)_1 - (ABC)_0$$

δ: $64\hat{\delta}=Y_{....1.}-Y_{....0.}=(D)_1-(D)_0$

$\alpha\delta$: $64\widehat{\alpha\delta}=Y_{..1..1.}+Y_{..0..0.}-Y_{..1..0.}-Y_{..0..1.}=(AD)_0-(AD)_1$

$\beta\delta$: $64\widehat{\beta\delta}=Y_{...1.1.}+Y_{...0.0.}-Y_{...1.0.}-Y_{...0.1.}=(BD)_0-(BD)_1$

$\alpha\beta\delta$: $64\widehat{\alpha\beta\delta}+8(\hat{\rho}_7+\hat{\rho}_8-\hat{\rho}_5-\hat{\rho}_6)=Y_{..10.0.}+Y_{..01.0.}+Y_{..00.1.}+Y_{..11.1.}-Y_{..00.0.}-Y_{..11.0.}$
$-Y_{..01.1.}-Y_{..10.1.}=(ABD)_1-(ABD)_0$

$\gamma\delta$: $64\widehat{\gamma\delta}=Y_{....11.}+Y_{....00.}-Y_{....10.}-Y_{....01.}=(CD)_0-(CD)_1$

$\alpha\gamma\delta$: $64\widehat{\alpha\gamma\delta}+8(\hat{\lambda}_6+\hat{\lambda}_8-\hat{\lambda}_5-\hat{\lambda}_7)=Y_{..1.00.}+Y_{..0.10.}+Y_{..0.01.}+Y_{..1.11.}-Y_{..0.00.}-Y_{..0.11.}$
$-Y_{..1.10.}-Y_{..1.01.}=(ACD)_1-(ACD)_0$

$\beta\gamma\delta$: $64\widehat{\beta\gamma\delta}+8(\hat{\lambda}_2+\hat{\lambda}_4-\hat{\lambda}_1-\hat{\lambda}_3)=Y_{...100.}+Y_{...101.}+Y_{...001.}+Y_{...111.}-Y_{...000.}-Y_{...011.}$
$-Y_{...101.}-Y_{...110.}=(BCD)_1-(BCD)_0$

$\alpha\beta\gamma\delta$: $64\widehat{\alpha\beta\gamma\delta}=(ABCD)_0-(ABCD)_1$

υ: $64\hat{\upsilon}=Y_{.....1}-Y_{.....0}=(E)_1-(E)_0$

$\alpha\upsilon$: $64\widehat{\alpha\upsilon}=Y_{..1...1}+Y_{..0...0}-Y_{..1...0}-Y_{..0...1}=(AE)_0-(AE)_1$

$\beta\upsilon$: $64\widehat{\beta\upsilon}=Y_{...1..1}+Y_{...0..0}-Y_{...1..0}-Y_{...0..1}=(BE)_0-(BE)_1$

$\alpha\beta\upsilon$: $64\widehat{\alpha\beta\upsilon}=(ABE)_1-(ABE)_0$

$\gamma\upsilon$: $64\widehat{\gamma\upsilon}=Y_{....1.1}+Y_{....0.0}-Y_{....1.0}-Y_{....0.1}=(CE)_0-(CE)_1$

$\alpha\gamma\upsilon$: $64\widehat{\alpha\gamma\upsilon}+8(\hat{\lambda}_3+\hat{\lambda}_4-\hat{\lambda}_1-\hat{\lambda}_2)=(ACE)_1-(ACE)_0$

$\beta\gamma\upsilon$: $64\widehat{\beta\gamma\upsilon}=8(\hat{\rho}_6+\hat{\rho}_8-\hat{\rho}_5-\hat{\rho}_7)=(BCE)_1-(BCE)_0$

$\alpha\beta\gamma\upsilon$: $64\widehat{\alpha\beta\gamma\upsilon}+8(\hat{\lambda}_5+\hat{\lambda}_6-\hat{\lambda}_7-\hat{\lambda}_8)=(ABCE)_0-(ABCE)_1$

$\delta\upsilon$: $64\widehat{\delta\upsilon}=Y_{.....11}+Y_{.....00}-Y_{.....10}-Y_{.....01}=(DE)_0-(DE)_1$

$\alpha\delta\upsilon$: $64\widehat{\alpha\delta\upsilon}+8(\hat{\rho}_4+\hat{\rho}_2-\hat{\rho}_1-\hat{\rho}_3)=(ADE)_1-(ADE)_0$

$\beta\delta\upsilon$: $64\widehat{\beta\delta\upsilon}+8(\hat{\lambda}_6+\hat{\lambda}_7-\hat{\lambda}_5-\hat{\lambda}_8)=(BDE)_1-(BDE)_0$

$\alpha\beta\delta\upsilon$: $64\widehat{\alpha\beta\delta\upsilon}+8(\hat{\lambda}_1+\hat{\lambda}_4-\hat{\lambda}_2-\hat{\lambda}_3)=(ABDE)_0-(ABDE)_1$

$\gamma\delta\upsilon$: $64\widehat{\gamma\delta\upsilon}=(CDE)_1-(CDE)_0$

$\alpha\gamma\delta\upsilon$: $64\widehat{\alpha\gamma\delta\upsilon}+8(\hat{\rho}_5+\hat{\rho}_8-\hat{\rho}_6-\hat{\rho}_7)=(ACDE)_0-(ACDE)_1$

$\beta\gamma\delta\upsilon$: $64\widehat{\beta\gamma\delta\upsilon}+8(\hat{\rho}_1+\hat{\rho}_4-\hat{\rho}_2-\hat{\rho}_3)=(BCDE)_0-(BCDE)_1$

$\alpha\beta\gamma\delta\upsilon$: $64\widehat{\alpha\beta\gamma\delta\upsilon}=(ABCDE)_1-(ABCDE)_0$

In the above equations, the $\hat{\rho}_f$ and $\hat{\lambda}_g$ effects are eliminated rather easily. For example, in the second from last equation above, substitution for $8\hat{\rho}_1+8\hat{\rho}_4-8\hat{\rho}_2-8\hat{\rho}_3$ from the first 4 equations for the $\hat{\rho}_f$, all effects are eliminated except $\widehat{\beta\gamma\delta\upsilon}$ and its coefficient is reduced to 32. The equation for $\beta\gamma\delta\upsilon$ now becomes:

$$32\widehat{\beta\gamma\delta\upsilon}=(BCDE)_0-(BCDE)_1-Y_{1\ldots\ldots}-Y_{4\ldots\ldots}+Y_{2\ldots\ldots}+Y_{3\ldots\ldots}=(BCDE)_0'-(BCDE)_1' \text{ from rows 5 to 8}$$

The remaining effects which are partially confounded with either rows or columns are obtained in the same manner as described above. The coefficient of each of these effects is 32 whereas the coefficient of each of the uncounfounded effects is 64. Also, the effect $\hat{\alpha}$, say, is equal to one-half the effect of nitrogen, or A = effect of nitrogen = $2\hat{\alpha}$. The same relationship holds for all other effects. (The notation and definition of effects is that used by Federer, loc. cited.)

The variance of the unconfounded effects is:

$$V(2\hat{\alpha})=V(A)=V\left(\frac{(A)_1-(A)_0}{2(16)}\right)=\frac{\sigma^2}{16}$$

The variance is estimated as (error mean square)/16. The variance of the partially confounded effects is

$$V(2\widehat{\alpha\beta\gamma})=V(ABC')=\frac{\sigma^2}{8}$$

and is estimated by (error mean square)/8. With these estimated variances confidence intervals may now be constructed after the error rate base (i.e., comparison-wise, experimentwise, etc.) has been selected.

The variance of the linear effect of potash or phosphorous is obtained as follows. The adjusted treatment total in terms of the effects may be obtained as (see Federer, loc. cited):

$$Y'_{\cdot\cdot hijkm}=\frac{1}{16}\Big\{(A)_h+(B)_i+(AB)_{h+i}+(C)_j+(AC)_{h+j}+(BC)_{i+j}+2(ABC')_{h+i+j}$$
$$+(D)_k+(AD)_{h+k}+(BD)_{i+k}+2(ABD')_{h+i+k}+(CD)_{j+k}+2(ACD')_{h+j+k}$$
$$+2(BCD')_{i+j+k}+(ABCD)_{h+i+j+k}+(E)_m+(AE)_{h+m}+(BE)_{i+m}+(ABE)_{h+i+m}$$
$$+(CE)_{j+m}+2(ACE')_{h+j+m}+2(BCE')_{i+j+m}+2(ABCE')_{h+i+j+m}+(DE)_{k+m}$$
$$+2(ADE')_{h+k+m}+2(BDE')_{i+k+m}+2(ABDE')_{h+i+k+m}+(CDE)_{j+k+m}$$
$$+2(ACDE')_{h+j+k+m}+2(BCDE')_{i+j+k+m}+(ABCDE)_{h+i+j+k+m}-15Y_{\cdots\cdots}\Big\}$$

where the primed effects are obtained from the rows or columns in which the effect is unconfounded with rows or columns. Summing over the subscripts hij the following four totals in terms of levels of effects are:

$$Y'_{\cdots\cdot 00}=\frac{1}{2}\left\{(D)_0+(E)_0+(DE)_0\right\}=Y_{\cdots\cdot 00}$$
$$Y'_{\cdots\cdot 01}=\frac{1}{2}\left\{(D)_0+(E)_1+(DE)_1\right\}=Y_{\cdots\cdot 01}$$
$$Y'_{\cdots\cdot 10}=\frac{1}{2}\left\{(D)_1+(E)_0+(DE)_1\right\}=Y_{\cdots\cdot 10}$$
$$Y'_{\cdots\cdot 11}=\frac{1}{2}\left\{(D)_1+(E)_1+(DE)_0\right\}=Y_{\cdots\cdot 11},$$

where the primed value is obtained from adjusted totals and the unprimed value from unadjusted totals. In this case the two sets of totals are equal.

The linear effect of phosphorous for equally spaced increasing levels of phosphorous denoted as 00, 01, 10, and 11 is

$$(P)_L=-3Y_{\cdots\cdot 00}-Y_{\cdots\cdot 01}+Y_{\cdots\cdot 10}+3Y_{\cdots\cdot 11}$$
$$=2(D)_1-2(D)_0+(E)_1-(E)_0\ .$$

The variance of $(P)_L/32=P_L$ is equal to $V(P_L)=5\sigma^2/16$.

The cubic effect of four equally spaced increasing levels of phosphorous denoted as 00, 01, 10, and 11 is

$$(P)_C = -Y'_{\cdot\cdot\cdot\cdot\cdot 00} + 3Y'_{\cdot\cdot\cdot\cdot\cdot 01} - 3Y'_{\cdot\cdot\cdot\cdot\cdot 10} + Y'_{\cdot\cdot\cdot\cdot\cdot 11}$$

$$= 2(E)_1 - 2(E)_0 - (D)_1 + (D)_0 \ ,$$

and the variance of $(P)_C/32 = P_C$ is

$$V(P_C) = 5\sigma^2/16 \ .$$

The linear and cubic contrasts for four equally spaced levels of potash are

$$(K)_L = -3Y_{\cdot\cdot\cdot 00\cdot\cdot} - Y_{\cdot\cdot\cdot 01\cdot\cdot} + Y_{\cdot\cdot\cdot 10\cdot\cdot} + 3Y_{\cdot\cdot\cdot 11\cdot\cdot}$$

$$(K)_C = -Y_{\cdot\cdot\cdot 00\cdot\cdot} + 3Y_{\cdot\cdot\cdot 01\cdot\cdot} - 3Y_{\cdot\cdot\cdot 10\cdot\cdot} + Y_{\cdot\cdot\cdot 11\cdot\cdot}$$

with variances

$$V\left(\frac{(K)_L}{32} = K_L\right) = \frac{5}{16}\,\sigma^2 = V\left(\frac{(K)_C}{32} = K_C\right) \ .$$

For contrasts of the form $K_L x P_L$ the individual variances will need to be computed as described from the $Y'_{\cdot\cdot hijkm}$; the variances for some of these will not be so simple as the variances obtained above, but are obtainable directly from the variances of the effects as described on page 8.