The Ghosh-Pratt Identity

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Abstract

A review of the Ghosh-Pratt Identity, which links the probability of false coverage and expected length in confidence set estimation

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The Ghosh-Pratt identity, independently discovered by Ghosh (1961) and Pratt (1961, connects two fundamental performance measures in confidence set estimation.

If we observe $X = x$, where $X \sim f(x|\theta)$, a *confidence set* for θ is a set $C(x)$ in the parameter space Θ. Associated with this confidence set is its *probability of coverage* $P_\theta[\theta \in C(X)]$, the probability of covering the true parameter. When constructing a confidence it is customary to specify a value for the probability of coverage, say $1-\alpha$, and then try to optimize another measure of performance among all confidence sets with coverage probability at least $1-\alpha$.

Perhaps the most natural measure of performance is the volume of the set (or length, if the set is an interval). Another measure, somewhat less obvious, but arising naturally through the confidence set/hypothesis testing relationship, is the *probability of false coverage*, $P_\theta[\theta' \in C(X)]$, $\theta \neq \theta'$, the probability of covering the value θ' when θ is the true parameter. The Ghosh-Pratt Identity establishes the connection between these two measures.

Ghosh-Pratt Identity. For a confidence set $C(x)$ with finite expected volume,

$$E_{\theta_0}\{\text{volume}[C(X)]\} = \int_\Theta P_{\theta_0}[\theta \in C(X)]\, d\theta,$$

that is, the expected volume at any parameter value θ_0 is equal to the integrated probability of false coverage.

Proof. The proof is quite elegant, following from an application of Fubini's Theorem to interchange the order of integration. The expected value is

$$\begin{aligned}
E_{\theta_0}\{\text{volume}[C(X)]\} &= \int_{\mathcal{X}} \{\text{volume[C(x)]}\}\, f(x|\theta_0)\, dx \\
(1) \qquad &= \int_{\mathcal{X}} \left\{ \int_{C(x)} d\theta \right\} f(x|\theta_0)\, dx \\
&= \int_\Theta \left\{ \int_{\{x:\theta\in C(x)\}} f(x|\theta_0)\, dx \right\} d\theta \qquad \text{(Fubini)} \\
&= \int_\Theta P_{\theta_0}[\theta \in C(X)]\, d\theta,
\end{aligned}$$

establishing the identity. □

Technically, at $\theta = \theta_0$, the integrand on the right of (1) is the probability of true coverage. However, this one point can be deleted from the range of integration without changing the value of the integral.

The hypothesis testing/set estimation relationship is well-known (see, for example, Lehmann 1986, Sections 5.6 and 5.6). Start with the hypothesis test $H_0 : \theta = \theta'$ versus $H_1 : \theta \neq \theta'$, where H_0 is accepted if $x \in A(\theta')$, the *acceptance region.* Form the confidence set $C(x)$ using the relation

$$\theta' \in C(x) \quad \Leftrightarrow \quad x \in A(\theta'). \tag{2}$$

Then if $A(\theta')$ results in a size α test of H_0, $C(x)$ is a $1 - \alpha$ confidence set.

Using (2), we can write (1) as

$$E_{\theta_0}\{\text{volume}[C(X)]\} = \int_{\Theta} P_{\theta_0}[X \in A(\theta')]\, d\theta', \tag{3}$$

so the expected volume is equal to an integrated probability of type II errors, the probabilities of accepting the false $H_0 : \theta = \theta'$ when the true value of the parameter is $\theta = \theta_0$.

Expression (3) gives a most useful form of the Ghosh-Pratt identity because, in a number of situations, it is easier to establish optimality of a family of tests than that of a confidence set. In particular, the theory of *uniformly most powerful* (UMP) and *uniformly most powerful unbiased* (UMPU) tests is well known (Lehmann 1986, Chapters 3 and 4). Starting with a family of optimal tests we can, using (2) and (3), obtain confidence sets of shortest expected length. For example, since the usual Student's t-test is UMPU, (where *unbiased* means that the power is always greater than the size), it follows that the usual Student's t-interval has shortest expected length among all unbiased intervals (intervals whose false coverage is less than $1 - \alpha$.).

Interestingly, in the case of one-sided confidence sets, the optimality relationship (3) changes somewhat. Although (1) and (3) remain true for one-sided intervals with finite expected length, the optimality of the type II errors of the UMP tests does not hold for the entire range of parameters, but only for parameter values on one side of the true parameter. For example, if we have a UMP test of $H_0 : \theta = \theta'$ versus $H_1 : \theta < \theta'$, the associated one-sided confidence interval $(-\infty, U(x)]$ minimizes $P_\theta[\theta' \leq U(X)]$ only for $\theta < \theta'$. Since (3) requires integration over the entire range of θ, one-sided test optimality does not immediately translate into expected length optimality. If we instead measure the size of the interval by the quantity $E_{\theta_0}\{U(X) - \theta_0\}$, the expected amount by which $U(X)$ overestimates the true θ_0. Then

$$E_{\theta_0}\{U(X) - \theta_0\} = \int_{\theta_0}^{\infty} P_{\theta_0}[\theta' < U(X)]\, d\theta', \tag{4}$$

and hence a one-sided interval formed from a family of UMP tests will minimize the left side of (4). This quantity was denoted the *expected excess* by Madansky (1962), who also exhibited a confidence interval with shorter expected length than that of the one based on inverting UMP tests.

Other applications of the Ghosh-Pratt Identity are found in the work of Cohen and Strawderman (1973), who used it to establish admissibility results for confidence regions. More recently, Brown, Casella and Hwang (1995) applied the identity to obtain a class of minimum volume confidence sets having application in the problem of determining bioequivalence, and Tseng (1994) used a similar construction to produce improved confidence sets for a multivariate normal mean.

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