

On the Existence of Stable Pairing Distributions

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1. Introduction

According to the American *Heritage Dictionary of the English Language* (3rd. ed., 1992) demography is the “the study of the characteristics of human populations such as size, growth, density, and vital statistics.” Hence it is not surprising that demography has, as an important theoretical component, the field of mathematical demography.

An important aspect of mathematical (which usually includes statistical) demography deals with the development of improved procedures for constructing life tables (invented by the famous astronomer Halley in 1693; as cited in Smith and Keyfitz 1977). As Keyfitz (1977) puts it the “...life table is a scheme for expressing the facts of mortality in terms of probabilities.” Milne (1815; as cited in Smith and Keyfitz 1977) adds that “they show the proportion of the people of each sex in every interval of age...”. Euler (1760; as cited in Smith and Keyfitz 1977) derived a set of formulas that defined the concept of stable age distribution, that is, a distribution

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in which the fraction of individuals in each age class remains fixed. He introduced the concept of stable age distribution under the assumptions that the per-capita birth rate is constant while the probabilities of survival remain constant for each age class. Euler used his stable population theory to input missing data. Smith and Keyfitz (1977) stress the importance of this theory by noting that “Euler’s virtually unknown article...anticipates important aspects of modern stable population theory for a one-sex population closed to migration.”

The connection between stable theory and population dynamics was the result of the efforts of Lotka (1907, 1922) and Sharpe and Lotka (1911). The Lotka/Sharpe results were put on a solid mathematical foundation (renewal theory) by Feller (1941). Feller’s work continues to generate research activity (see for example, Inaba 1988 and references therein).

One concern of renewal theory is the study of the dynamics of one-sex populations. Our article deals with the study of two-sex populations and their associated stable distributions for populations structured according to fixed characteristics. We illustrate our results with two examples. The structure of these two examples allows for a full analysis a (existence, uniqueness, and stability). The formulation of age-structured models is straightforward (see Castillo-Chavez et al. 1991), but we do not study them because their mathematical analysis is difficult (see Hadeler 1989a, b; Hadeler 1992; Hadeler et al. 1988).

This article is organized as follows: Section 2 briefly reviews the literature on two-sex models and introduces the pair-formation model of Kendall/Keyfitz and Dietz/Hadeler for a homogeneously mixing population as well as the logistic pair-formation model of Castillo-Chavez and Huang (1994). Section 3 partially generalizes the logistic model of Castillo-Chavez and Huang (1994) by incorporating nonrandom mixing in a population structured by fixed characteristics, that is, we move from a single one-group two-sex model to an n -group two-sex multigroup heterogeneously mixing model. Section 4 partially analyzes this general n -group

model when all partnerships have the same average duration. Section 5 addresses the problem of uniqueness for stable pair distributions through the discussion of two examples when the total population is bounded. Section 6 introduces a nonlinear n -group pair-formation model that generalizes the Kendall/Keyfitz and Dietz/Hadeler one-group model; a multigroup model is formulated in terms of a homogeneous system of order one. Sections 7 and 8 address the problems of existence and stability (trajectory) of exponential solutions.

2. Two-Sex Models

The study of pair-formation or two-sex models in demography was pioneered by Kendall (1949) and Keyfitz (1949). Recent extensions are found in the articles by Fredrickson (1971), McFarland (1972), Parlett (1972), and Pollard (1973). Attempts to understand the basic mechanisms behind the transmission dynamics of the HIV/AIDS epidemic created additional interest in the study of pair-formation models. Increased efforts to understand the dynamics of sexually-transmitted diseases (STD's) have added emphasis to the study of models with partnerships of variable duration (models with multiple simultaneous partnerships are poorly understood, but see Dietz et. al. 1991). Classifications are not usually accurate; however, we roughly identify two theoretical directions in the study of structured populations in demography: the first deals with the study of contact models (a way of studying the effects of high rates of partnership change) while the second deals explicitly with the dynamics of couples (pair-formation models). Both approaches are an attempt to incorporate diverse and complex *social dynamics* into models for disease spread. Luo and Castillo-Chavez (1991) show that as the rates of pair-dissolution approach infinity (that is, as the duration of partnerships approaches zero) pair-formation (and epidemic stochastic) models reduce to contact models (including the general epidemic model of Bailey, see Bailey, 1975) with identical pa-

rameters. Luo and Castillo-Chavez (1991) based their results on the contact/mixing framework developed by Blythe and Castillo-Chavez (1989), Castillo-Chavez and Blythe (1989), Busenberg and Castillo-Chavez (1989, 1991), and Castillo-Chavez and Busenberg (1991). Which model to use and when to use it depends on the type of questions asked and the available data. However, both can be studied under the same theoretical framework.

Extensive connections between social dynamics, demography, social networks, and epidemiology have been encountered over the last few years (see Anderson et al. 1989; Blythe and Castillo-Chavez 1989; Blythe et al. 1991, 1992; Busenberg and Castillo-Chavez 1989, 1991; Castillo-Chavez 1989; Castillo-Chavez et al. 1991; Castillo-Chavez et al. 1994; Dietz 1988; Dietz and Hadeler 1988; Gupta et al. 1989; Hadeler 1989a, b; Hadeler and Nagoma 1990; Huang et al. 1992; Hyman and Stanley 1989; Jacquez et al. 1988, 1989; May and Anderson 1989; Sattenspiel 1987a, b; Sattenspiel and Simon 1988; Sattenspiel and Castillo-Chavez 1990; Waldst tter 1989). Most of these studies (but not all) involve only one-sex populations. In this article, we are concerned exclusively with two-sex models for multigroup populations. We discuss models that support exponential solutions (homogeneous systems of order one) as well as models that only support bounded solutions.

First we introduce the Kendall/Keyfitz and Dietz/Hadeler model (homogeneous system of order one) and include a summary of its properties. Let $m_s(t)$ denote the population of single males at time t , $f_s(t)$ the population of single females at time t , and $p(t)$ the population of paired individuals (heterosexual pairings only) at time t . Furthermore, let μ_m and μ_f denote the per-capita removal rates due to death or other factors for males and females; let β_m , β_f denote the per-capita birth rates for males and females respectively; and let σ denote the per-pair separation rate. With these definitions, one easily derives the following demographic pair-formation model:

$$\begin{aligned}
\dot{m}_s &= -\mu_m m_s + (\beta_m + \mu_f + \sigma)p - \phi(m_s, f_s) \\
\dot{f}_s &= -\mu_f f_s + (\beta_f + \mu_m + \sigma)p - \phi(m_s, f_s) \\
\dot{p} &= -(\mu_m + \mu_f + \sigma)p + \phi(m_s, f_s),
\end{aligned} \tag{2.1}$$

where ϕ denotes the pair-formation rate.

It is assumed that ϕ is differentiable for $(m_s, f_s) \in \text{IR}_+^2 \setminus \{(0, 0)\}$ and that it satisfies the properties:

$$\begin{aligned}
\phi(m, f) &\geq 0, \\
\phi(m + u, f + v) &\geq \phi(m, f), \\
\phi(\alpha m, \alpha f) &= \alpha \phi(m, f), \\
\phi(m, 0) = \phi(0, f) &= 0.
\end{aligned} \tag{PF}$$

(See Fredrickson 1971, Hoppensteadt 1975, Dietz 1988, Dietz and Hadeler 1988, Castillo-Chavez et. al. 1994, and references therein.)

System (2.1) and (PF) give a homogeneous system of degree one that supports exponential solutions. Hadeler and collaborators (Hadeler 1989a, b; Hadeler and Nagoma 1990; Hadeler et al. 1988; Hadeler 1992) have developed an extensive theory for homogeneous systems of this type. Using this theory, the following results for system (2.1) have been verified (see Dietz and Hadeler 1988, Waldstätter 1989, and references therein):

I. System (2.1) always has two exponential solutions

$$m_s(t) = e^{-\mu_m t}, \quad f_s(t) = p(t) \equiv 0, \tag{a}$$

and

$$f_s(t) = e^{-\mu_f t}, \quad m_s(t) = p(t) \equiv 0. \tag{b}$$

II. If ϕ_m and ϕ_f denote the partial derivatives of ϕ with respect to the first and the second variable of ϕ respectively and if

$$\mu_m < \mu_f - \frac{\beta_f \phi_f(1, 0)}{\mu_f + \sigma + \phi_f(1, 0)},$$

then there is no strictly positive exponential solution, and the exponential solution in *I.(a)* is trajectorally stable (see Hadeler's references for this definition of stability), while the solution expressed in *I.(b)* is trajectorally unstable.

III. If

$$\mu_f < \mu_m - \frac{\beta_m \phi_m(0, 1)}{\mu_m + \sigma + \phi_m(0, 1)},$$

then there is no strictly positive exponential solution. The exponential solution *I.(b)* is stable while the exponential solution in *I.(a)* is unstable.

IV. If

$$\mu_m > \mu_f - \frac{\beta_f \phi_f(1, 0)}{\mu_f + \sigma + \phi_f(1, 0)}$$

and

$$\mu_f > \mu_m - \frac{\beta_m \phi_m(0, 1)}{\mu_m + \sigma + \phi_m(0, 1)}$$

then (2.1) has a unique strictly positive exponential solution, which is stable, while the solutions in *I.(a)* and *I.(b)* are both unstable.

Models that also incorporate nonlinear birth and separation processes for two-sex homogeneously mixing populations are also being studied. To contrast the effects of mathematical homogeneity of order one, we revisit the extension of the system (2.1) introduced by Castillo-Chavez and Huang (1994). This extension incorporates nonlinear birth and dissolution rates; in fact, it assumes that the birth rate $\beta = \beta(T)$ and the separation rate $\sigma = \sigma(T)$ are functions of the total population $T = m_s + f_s + 2p$. They also introduce the constants γ and $1 - \gamma$ ($0 < \gamma < 1$), representing the proportions of the male and female birth rates respectively. They arrive at the system:

$$\begin{aligned} \dot{m}_s &= -\mu_m m_s + [\gamma \beta(T) + \mu_f + \sigma(T)]p - \phi(m_s, f_s) \\ \dot{f}_s &= -\mu_f f_s + [(1 - \gamma)\beta(T) + \mu_m + \sigma(T)]p - \phi(m_s, f_s) \\ \dot{p} &= -[\mu_m + \mu_f + \sigma(T)]p + \phi(m_s, f_s), \end{aligned} \tag{2.2}$$

where β and σ are assumed to satisfy the following properties:

$$\frac{d\beta(T)}{dT} < 0, \quad \lim_{T \rightarrow \infty} \beta(T) = 0. \tag{H1}$$

$$\frac{d\sigma(T)}{dT} \geq 0. \quad (\text{H2})$$

Castillo-Chavez and Huang (1994) have established the following results:

Result A *If either*

$$\min \left\{ \frac{\gamma\beta(0)}{\mu_m}, \frac{(1-\gamma)\beta(0)}{\mu_f} \right\} \leq 1 \quad (2.3)$$

or

$$\min \left\{ \frac{\gamma\beta(0)}{\mu_m}, \frac{(1-\gamma)\beta(0)}{\mu_f} \right\} > 1 \quad (2.4)_a$$

and

$$\mu_m + \mu_f + \sigma(0) \geq \phi \left(\frac{\gamma\beta(0)}{\mu_m} - 1, \frac{(1-\gamma)\beta(0)}{\mu_m} - 1 \right), \quad (2.4)_b,$$

then for any given initial condition $z = (m_0, f_0, p_0) \in \Omega$

$$\lim_{t \rightarrow \infty} m(t, z) = \lim_{t \rightarrow \infty} f(t, z) = \lim_{t \rightarrow \infty} p(t, z) = 0.$$

Condition (2.3) implies that either the male growth rate or the female growth rate is negative, that is, either the total male population or the total female population vanishes and, consequently, so does the total paired population. From Condition (2.4)_a it is not obvious that the population goes to zero because both maximum male and female reproductive numbers, $\frac{\gamma\beta(0)}{\mu_m}$ and $\frac{(1-\gamma)\beta(0)}{\mu_f}$, exceed one. However, Condition (2.4)_b roughly implies that the growth rate for paired-individuals is negative and, therefore, the reproduction rates of male and female populations are sufficiently reduced so that extinction takes place.

Result B *For system (2.2), if*

$$\begin{aligned} \min \left\{ \frac{\gamma\beta(0)}{\mu_m}, \frac{(1-\gamma)\beta(0)}{\mu_f} \right\} &> 1, \\ \mu_m + \mu_f + \sigma(0) &< \phi \left(\frac{\gamma\beta(0)}{\mu_m} - 1, \frac{(1-\gamma)\beta(0)}{\mu_m} - 1 \right). \end{aligned} \quad (2.5)$$

then the zero solution is unstable and there exists a unique positive equilibrium (m^*, f^*, p^*) . Furthermore, the unique positive equilibrium is locally asymptotically stable.

If the total population T is small, then Condition (3.5) roughly implies that the average growth rates of male, female, and paired populations are positive. Under these circumstances, the total population persists and we expect the existence of at least one positive steady state. Generalizations of models (2.2) and (2.1) to heterogeneously mixing populations are the subject of further analysis in Sections 3 and 6 respectively.

3. Multiple Group Pair-Formation Model

The first general mathematical study of single group, two-sex pair-formation models was carried out by Hadeler and Nagoma (1990) and Hadeler et al. (1988). This analysis was extended to handle models with time delay effects (for example of models derived from pair formation models with age-structure, see Hadeler 1992). However, there was no progress for multiple-group pair-formation models until the recent work of Lubkin and Castillo-Chavez (1993) and Lubkin et al. (1993).

The deterministic two-sex population pair-formation model with n subgroups that we formulate in this section generalizes *some* of the models of Lubkin and Castillo-Chavez (1993). Our modeling approach combines a property similar to **(PF)** with the two-sex mixing framework of Castillo-Chavez and Busenberg (1991). We assume that the population under consideration can be stratified into n groups. These groups are defined through the use of assumed fixed characteristics such as race, gender, social status, etc. The following definitions are required for the formulation of this model. Let

$$m_i = \text{number of single males in group } i,$$

- f_i = number of single females in group i ,
 P_{ij} = number of pairs with type i male and type j female,
 μ_m = per-capita male mortality rate,
 μ_f = per-capita female mortality rate,
 Λ_i^m = recruitment rate of single males of type i ,
 Λ_i^f = recruitment rate of single females of type i ,
 σ_{ij} = per- ij -pair separation rate,
 C_i = per capita pair formation rate of type i males,
 B_i = per capita pair formation rate of type i females,
 p_{ij} = proportion of type- i male pairings which are with j females, given that they paired,
 q_{ij} = proportion of type- i female pairings which are with j males, given that they paired.

The deterministic dynamics of a two-sex n -group pair-formation model can be described with the above definitions by the following system of differential equations:

$$\begin{aligned}\dot{m}_i &= \Lambda_i^m - (\mu_m + C_i(m, f))m_i + \sum_j (\mu_f + \sigma_{ij})P_{ij}, \\ \dot{f}_i &= \Lambda_i^f - (\mu_f + B_i(m, f))f_i + \sum_j (\mu_m + \sigma_{ji})P_{ji}, \\ \dot{P}_{ij} &= C_i(m, f)m_i p_{ij} - (\mu_m + \mu_f + \sigma_{ij})P_{ij},\end{aligned}\tag{3.1}$$

where

$$m = (m_1, \dots, m_n)^T, \quad f = (f_1, \dots, f_n)^T.$$

Since $\sum_i^n C_i(m, f)m_i$ and $\sum_i^n B_i(m, f)f_i$ represent the total pair-formation rates of males and females respectively, we must have

$$(i) \quad \sum_i^n C_i(m, f)m_i = \sum_i^n B_i(m, f)f_i.$$

From the definitions of p_{ij} and q_{ij} we must also have

$$(ii) \quad \sum_j^n p_{ij} = \sum_j^n q_{ij} = 1,$$

and

$$(iii) \quad C_i(m, f)m_i p_{ij} = B_j(m, f)f_j q_{ji}.$$

Conditions (i), (ii) and (iii) are known as the Mixing Axioms or Mixing Properties (see Castillo-Chavez and Busenberg 1991, Castillo-Chavez et al. 1994, Hsu Schmitz et al. 1993, 1994, and Hsu 1994). Solutions to these axioms have been studied by Castillo-Chavez and Busenberg (1991). They are able to characterize all mixing matrices p'_{ij} s and q'_{ij} s in terms of B_i , C_i , m_i , and f_i . They also show that the only separable solution p'_{ij} s and q'_{ij} s to the above properties corresponds to heterogeneous random or proportionate mixing (the so-called Ross Solution; see Castillo-Chavez and Busenberg 1991). Their characterization of all solutions is given in terms of multiplicative perturbations of Ross solutions (they are named after Ross because its implicit relationship to the mixing axioms, see Ross 1911). Refinements of these results can be found in Hsu Schmitz et al. (1994), and Hsu Schmitz (1994).

Furthermore, we must still deal with the problem of characterizing the nonlinear pair-formation rates $B_i(m, f)f_i$ and $C_i(m, f)m_i$ which in general are complex functions of their arguments. The following required properties for the B'_i s and the C_i seem feasible in general, and in fact, they do not contradict property **(PF)** in the one group case:

$$\begin{aligned} \frac{\partial(C_i(m, f)m_i)}{\partial m_i} &\geq 0, & \frac{\partial(B_i(m, f)f_i)}{\partial f_i} &\geq 0, & i &= 1, \dots, n. \\ \frac{\partial C_i(m, f)}{\partial m_j} &\leq 0, & \frac{\partial C_i(m, f)}{\partial f_i} &\geq 0, & j &= 1, \dots, n, \\ \frac{\partial B_i(m, f)}{\partial m_i} &\geq 0, & \frac{\partial B_i(m, f)}{\partial f_i} &\leq 0, & j &= 1, \dots, n. \end{aligned} \tag{H}$$

Thus, an increase of the type i female population will increase the total pair formation rate ($C_i(m, f)m_i$) for type i males, and an increase in the population of any

type of males/females will reduce/increase the per capita pair formation rate for males/females. Because (H) seems reasonable, we assume its validity throughout the rest of this article.

Finally, we observe that to model heterogeneous pair-formation, the group pair-formation rates $B_i(m, f)f_i$ and $C_i(m, f)m_i$ must be distributed per unit of time among the various groups under reasonable mixing/pairing constraints. Hence, we assume that the Mixing Axioms (i)-(iii) hold for the rest of this article. The modeling approach described in this article provides a framework for modeling various other types of biological interactions. Castillo-Chavez et al. (1994) and Blythe et al. (1994) report on applications of this framework to cultural anthropology, food web dynamics, vector-transmitted diseases, population genetics, and distribution theory.

This paper does not provide a complete analysis of the general pair-formation model (3.1). Section 4 discusses model (3.1), assuming constant recruitment rates and equal separation rates, while Sections 6-8 further relax these assumptions.

4. Pair-Formation Models with Constant Recruitment and Identical Average Duration

First we study a relatively simple sub-model of system (3.1). We suppose that the separation rates σ_{ij} are equal, that is,

$$\sigma_{ij} = \sigma. \quad i, j = 1, 2, \dots, n.$$

We also assume that the recruitments Λ_i^m , Λ_i^f , $i = 1, \dots, n$ are constant. Using these assumptions and the new variables

$$P_i = \sum_j P_{ij}, \quad Q_i = \sum_j P_{ji}, \quad i = 1, \dots, n,$$

we can rewrite System (3.1) in the equivalent form:

$$\begin{aligned}\dot{m}_i &= \Lambda_i^m - (\mu_m + C_i(m, f))m_i + (\mu_f + \sigma)P_i \\ \dot{f}_i &= \Lambda_i^f - (\mu_f + B_i(m, f))f_i + (\mu_m + \sigma)Q_i \\ \dot{P}_i &= C_i(m, f)m_i - (\mu_m + \mu_f + \sigma)P_i \\ \dot{Q}_i &= B_i(m, f)f_i - (\mu_m + \mu_f + \sigma)Q_i.\end{aligned}\tag{4.1}$$

From (4.1) we conclude immediately that

$$\frac{d}{dt}(m_i + P_i) = \Lambda_i^m - \mu_m(m_i + P_i),$$

and that

$$\frac{d}{dt}(f_i + Q_i) = \Lambda_i^f - \mu_f(f_i + Q_i).$$

Hence, we have

$$\lim_{t \rightarrow \infty} (m_i(t) + P_i(t)) = \frac{\Lambda_i^m}{\mu_m}$$

and

$$\lim_{t \rightarrow \infty} (f_i(t) + Q_i(t)) = \frac{\Lambda_i^f}{\mu_m}.$$

(**Note:** $m_i + P_i$ is the total population of type i males and $f_i + Q_i$ is the total population of type i females.)

If we now let

$$m_i + P_i = \frac{\Lambda_i^m}{\mu_m}, \quad f_i + Q_i = \frac{\Lambda_i^f}{\mu_m},\tag{4.2}$$

then Thieme's results on asymptotically autonomous systems (see Thieme 1992, 1993a,b; Castillo-Chavez and Thieme 1993) make it possible to reduce the study of the dynamics of system (4.1) to the study of the dynamics of the following limiting system:

$$\begin{aligned}\dot{m}_i &= N_i^m - (\mu + C_i(m, f))m_i := G_i^m(m, f) \\ \dot{f}_i &= N_i^f - (\mu + B_i(m, f))f_i := G_i^f(m, f),\end{aligned}\tag{4.3}$$

where $\mu = \mu_m + \mu_f + \sigma$ and

$$N_i^m = \left(\frac{\mu_f + \sigma}{\mu_m} + 1 \right) \Lambda_i^m, \quad N_i^f = \left(\frac{\mu_m + \sigma}{\mu_f} + 1 \right) \Lambda_i^f.$$

Let

$$\Omega = \left\{ (m, f) \in \mathbb{R}_+^{2n}; \quad 0 \leq m_i \leq \frac{N_i^m}{\mu}, \quad 0 \leq f_i \leq \frac{N_i^f}{\mu}, \quad i = 1, \dots, n \right\}$$

then Ω is positively invariant to the flow of (4.3) and any solution with initial value on the boundary of Ω will eventually enter inside of Ω . Hence, there is at least one equilibrium in the interior of Ω . This is the content of the following theorem:

Theorem 4.1 *If (4.3) has only one equilibrium in Ω , then it is a global attractor of Ω .*

Proof. Let $G^m = (G_1^m, \dots, G_n^m)^T$, $G^f = (G_1^f, \dots, G_n^f)^T$. Then by using the hypotheses (H), we can see that the Jacobian matrix of $(G^m, G^f)^T$ at (m, f) is of the form

$$\frac{\partial(G^m, G^f)}{\partial(m, f)} = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix},$$

where $W_{ij} \in \mathbb{R}^{n \times n}$, $(i, j = 1, 2)$ the off-diagonal entries of W_{11} , W_{22} are non-negative, and W_{21} , W_{12} are nonpositive matrices for $(m, f) \in \Omega$. Hence, System (4.3) generates a monotone flow. From the results on monotone flows of Smith and Thieme (1990) Theorem (4.1) follows.

Remark: Under hypotheses (H) it has not been possible to prove the existence of a unique positive equilibrium. In fact, we believe that multiple equilibria are possible. Nevertheless, Section 5 provides two examples where uniqueness can be established.

5. Uniqueness of Stable Paired Distributions: Two Examples

For the two examples of this section we are able to show the existence of a unique positive stable pairing distribution. These examples consider arbitrary mixing matrices but specific per-capita pairing rates. **Example 1** This example uses as functional forms for the per-capita pair-formation rates C_i and B_i , $i = 1, \dots, n$ the so-called harmonic means

$$\begin{aligned} C_i(m, f) &= \frac{\alpha_i \sum_k \beta_k f_k}{\sum_k \alpha_k m_k + \sum_k \beta_k f_k}, \\ B_i(m, f) &= \frac{\beta_i \sum_k \alpha_k m_k}{\sum_k \alpha_k m_k + \sum_k \beta_k f_k}, \end{aligned} \quad (5.1)$$

or

$$\begin{aligned} C_i(m, f) &= \alpha_i \sum_k \frac{\beta_k f_k}{\alpha_i m_i + \beta_k f_k}, \\ B_i(m, f) &= \beta_i \sum_k \frac{\alpha_k m_k}{\alpha_k m_k + \beta_i f_i} \end{aligned} \quad (5.1)^*$$

where $\alpha_i, \beta_i, i = 1, \dots, n$ are given positive constants. It is clearly that $C_i(m, f)$ and $B_i(m, f)$ satisfy the hypotheses (H).

Theorem 5.1 Let $C_i(m, f)$ and $B_i(m, f)$ be defined as in (5.1), then System (4.3) has a unique equilibrium in Ω .

Proof. $m^* = (m_1^*, \dots, m_n^*)^T, f^* = (f_1^*, \dots, f_n^*)^T$ is an equilibrium of (4.3) if and only if

$$\begin{aligned} N_i^m &= (\mu + \frac{\alpha_i \sum_k \beta_k f_k^*}{\sum_k \alpha_k m_k^* + \sum_k \beta_k f_k^*}) m_i^*, \\ N_i^f &= (\mu + \frac{\beta_i \sum_k \alpha_k f_k^*}{\sum_k \alpha_k m_k^* + \sum_k \beta_k f_k^*}) f_i^*. \end{aligned} \quad (5.2)$$

If we let

$$T_1 = \sum_k \alpha_k m_k^*, \quad T_2 = \sum_k \beta_k f_k^*, \quad (5.3)$$

and substitute (5.3) into (5.2), then we have

$$N_i^m = (\mu + \frac{\alpha_i T_2}{T_1 + T_2}) m_i^*, \quad N_i^f = (\mu + \frac{\beta_i T_1}{T_1 + T_2}) f_i^*, \quad i = 1, \dots, n. \quad (5.4)$$

From (5.4) it follows that

$$m_i^* = \frac{N_i^m(T_1 + T_2)}{\mu T_1 + (\mu + \alpha_i)T_2}, \quad f_i^* = \frac{N_i^f(T_1 + T_2)}{\mu T_2 + (\mu + \beta_i)T_1}. \quad (5.5)$$

Equations (5.3) and (5.5) yield that

$$\begin{aligned} \sum_k \frac{\alpha_k N_k^m(T_1 + T_2)}{\mu T_1 + (\mu + \alpha_k)T_2} - T_1 &= 0 \\ \sum_k \frac{\beta_k N_k^f(T_1 + T_2)}{\mu T_2 + (\mu + \beta_k)T_1} - T_2 &= 0. \end{aligned} \quad (5.6)$$

Conversely, if $T_1 > 0, T_2 > 0$ satisfies Equation (5.6), then setting m_i^*, f_i^* as defined in (5.5) gives a solution of (5.2). Therefore, it is enough to prove that (5.6) has a unique solution.

To study the question of the uniqueness of equilibrium to (5.6), we consider the system of differential equations

$$\begin{aligned} \dot{T}_1 &= \sum_k \frac{\alpha_k N_k^m(T_1 + T_2)}{\mu T_1 + (\mu + \alpha_k)T_2} - T_1 \\ \dot{T}_2 &= \sum_k \frac{\beta_k N_k^f(T_1 + T_2)}{\mu T_2 + (\mu + \beta_k)T_1} - T_2. \end{aligned} \quad (5.7)$$

System (5.7) has a positively invariant set,

$$\Sigma = \left\{ (T_1, T_2) \in \mathbb{R}_+^2; 0 \leq T_1 \leq \sum_k \frac{\alpha_k N_k^m}{\mu}, 0 \leq T_2 \leq \sum_k \frac{\beta_k N_k^f}{\mu} \right\},$$

which is an attractor of all points in \mathbb{R}_+^2 . Furthermore, a solution of (5.6) in \mathbb{R}_+^2 must be an equilibrium of (5.7) in Σ .

We claim that any equilibrium of (5.7) in Σ is asymptotically stable. In fact, if T_1^*, T_2^* is an equilibrium, then the Jacobian matrix at T_1^*, T_2^* is

$$J = \begin{bmatrix} -1 + \sum_k \frac{\alpha_k^2 N_k^m T_2^*}{[\mu T_1^* + (\mu + \alpha_k)T_2^*]^2} & - \sum_k \frac{\alpha_k^2 N_k^m T_1^*}{[\mu T_1^* + (\mu + \alpha_k)T_2^*]^2} \\ - \sum_k \frac{\beta_k^2 N_k^f T_2^*}{[\mu T_2^* + (\mu + \beta_k)T_1^*]^2} & -1 + \sum_k \frac{\beta_k^2 N_k^f T_1^*}{[\mu T_2^* + (\mu + \beta_k)T_1^*]^2} \end{bmatrix}.$$

Since

$$\begin{aligned}
\text{tr}(J) &= -2 + \sum_k \frac{\alpha_k^2 N_k^m T_2^*}{[\mu T_1^* + (\mu + \alpha_k) T_2^*]^2} + \sum_k \frac{\beta_k^2 N_k^f T_1^*}{[\mu T_2^* + (\mu + \beta_k) T_1^*]^2} \\
&= -2 + \sum_k \frac{\alpha_k N_k^m}{[\mu T_1^* + (\mu + \alpha_k) T_2^*]} \frac{\alpha_k T_2^*}{[\mu T_1^* + (\mu + \alpha_k) T_2^*]} \\
&\quad + \sum_k \frac{\beta_k N_k^f}{[\mu T_2^* + (\mu + \beta_k) T_1^*]} \frac{\beta_k T_1^*}{[\mu T_2^* + (\mu + \beta_k) T_1^*]} \\
&\leq -2 + \sum_k \frac{\alpha_k N_k^m}{[\mu T_1^* + (\mu + \alpha_k) T_2^*]} + \sum_k \frac{\beta_k N_i^f}{[\mu T_2^* + (\mu + \beta_k) T_1^*]} \\
&= -2 + \frac{T_1^*}{T_1^* + T_2^*} + \frac{T_1^*}{T_1^* + T_2^*} \\
&= -1 \\
&< 0,
\end{aligned}$$

and

$$\begin{aligned}
\det(J) &= (-1 + \sum_k \frac{\alpha_k^2 N_k^m T_2^*}{[\mu T_1^* + (\mu + \alpha_k) T_2^*]^2})(-1 + \sum_k \frac{\beta_k^2 N_k^f T_1^*}{[\mu T_2^* + (\mu + \beta_k) T_1^*]^2}) \\
&\quad - (\sum_k \frac{\alpha_k^2 N_k^m T_2^*}{[\mu T_1^* + (\mu + \alpha_k) T_2^*]^2})(\sum_k \frac{\beta_k^2 N_k^f T_1^*}{[\mu T_2^* + (\mu + \beta_k) T_1^*]^2}) \\
&= 1 - \sum_k \frac{\alpha_k^2 N_k^m T_2^*}{[\mu T_1^* + (\mu + \alpha_k) T_2^*]^2} - \sum_k \frac{\beta_k^2 N_k^f T_1^*}{[\mu T_2^* + (\mu + \beta_k) T_1^*]^2} \\
&> 1 - \sum_k \frac{\alpha_k N_k^m}{[\mu T_1^* + (\mu + \alpha_k) T_2^*]} - \sum_k \frac{\beta_k N_k^f}{[\mu T_2^* + (\mu + \beta_k) T_1^*]} \\
&= 0,
\end{aligned}$$

we conclude, from elementary ODE theory, that the equilibrium (T_1^*, T_2^*) is locally asymptotically stable. Furthermore, since (5.7) is a competitive system, there are no periodic solutions in Σ (Hirsch 1982). It follows from two-dimensional ODE theory (planar systems) that there is only one equilibrium in Σ . Hence, System (5.6) has a unique solution. This completes the proof.

Putting Theorem 4.1 and 5.1 together we conclude:

Theorem 5.2 Suppose $\sigma_{ij} = \sigma$, $i, j = 1, \dots, n$ and $C_i(m, f)$, $B_i(m, f)$ are given by (5.1), then the system (3.1) has a unique positive equilibrium which is a global attractor of all nonnegative solutions.

Remark 1 Theorem 5.2 also holds if $C_i(m, f)$ and $B_i(m, f)$ are given by (5.1)*. However, we omit the proof.

Example 2 Consider the case where *females choose their partners*, that is, males get chosen at a constant per-female pairing rate. We assume that the per-capita rate of pair formation of type i females is

$$B_i(m, f) = b_i = \text{constant}, \quad i = 1, \dots, n. \quad (5.8)$$

Hence, from Mixing Axiom (iii), it follows that $C_i(m, f)$ cannot be, in general, constant. A valid selection must incorporate, into the definition of the per-capita rate of pair-formation of type i males, the effects a variable partnership pool. A choice of a multiplier, that is, a selection that guarantees that the Mixing Axioms are satisfied, is given by the sexual-activity ratio. Explicitly, we let

$$C_i(m, f) = c_i \frac{\sum_{k=1}^n b_k f_k}{\sum_{k=1}^n c_k m_k}, \quad i = 1, \dots, n, \quad (5.9)$$

where c_1, \dots, c_n are positive constants. These per-capita pair-formation rates B_i and C_i do not contradict the Mixing Axioms and hypothesis **(H)**. However, we must be aware that $C_i(m, f)$ will be unbounded as $m \rightarrow 0$, that is, solutions could go to zero or even become negative. In order to avoid this situation, we add constraints to the selection of the constants b_i and c_i . Explicitly, we assume that the dominant eigenvalue of the matrix

$$M = \text{diag}(N_i^m) \begin{bmatrix} c_1 & \cdots & c_n \\ \cdot & \cdots & \cdot \\ c_1 & \cdots & c_n \end{bmatrix} - \text{diag}(c_i \sum_{k=1}^n b_k \frac{N_k^f}{\mu + b_k}) \quad (5.10)$$

is positive. Next, we show that Equation (4.3) has a unique positive equilibrium when B_i 's and C_i 's are defined by (5.8) - (5.9) and M has a positive dominant eigenvalue.

Since $B_i = b_i$ is a constant for each i then the equilibrium

$$(m^*, f^*) = (m_i^*, \dots, m_n^*, f_i^*, \dots, f_n^*)$$

must satisfy

$$f_i^* = \frac{N_i^f}{\mu + b_i}.$$

Moreover, m^* satisfies the system

$$N_i^m - (\mu + c_i \frac{\sum_{k=1}^n b_k f_k^*}{\sum_{k=1}^n c_k m_k}) m_i = 0, \quad i = 1, \dots, n,$$

or, more succinctly, the system

$$S(m) = 0,$$

where the i th component of S takes the form

$$S_i(m) = (N_i^m - \mu m_i) \sum_{k=1}^n c_k m_k - c_i \left(\sum_{k=1}^n b_k f_k^* \right) m_i. \quad i = 1, \dots, n. \quad (5.11)$$

The function S (from a well known result) has a strictly positive zero if and only if the dominant eigenvalue of

$$DS(0) = \left(\frac{\partial S_i(0)}{\partial m_j} \right)_{n \times n} = M$$

is positive. Consequently, we have established the following result:

Theorem 5.3 *Let $B_i(m, f)$ and $C_i(m, f)$ be defined by (5.8) - (5.9) and assume that the matrix in (5.10) has a positive dominant eigenvalue, then System (4.1) has a unique positive equilibrium which is globally asymptotically stable.*

A similar result is obtained if the roles of male and female are reversed.

Remark 2 When System (4.1) has a unique positive equilibrium we can also obtain a useful relationship between the asymptotic solutions and the asymptotic mixing matrices: *A stable pairing distribution*. The argument goes as follows: from system (3.1) we notice that P_{ij} , the number of pairs of type i male and type j females, satisfies the equation

$$\dot{P}_{ij} = C_i(m, f)m_i p_{ij} - (\mu_m + \mu_f + \sigma)P_{ij}.$$

Using the formula of variation of constants and letting $t \rightarrow \infty$, we obtain

$$\begin{aligned} P_{ij}(t) &= P_{ij}(0)e^{-(\mu_m + \mu_f + \sigma)t} \\ &\quad + \int_0^t e^{-(\mu_m + \mu_f + \sigma)(t-s)} C_i(m(s), f(s))m_i(s)p_{ij}(m(s), f(s))ds \\ &\rightarrow \frac{C_i(m^*, f^*)m_i^* p_{ij}(m^*, f^*)}{\mu_m + \mu_f + \sigma} \\ &= \frac{B_j(m^*, f^*)f_j^* q_{ji}(m^*, f^*)}{\mu_m + \mu_f + \sigma}, \end{aligned}$$

where (m_1^*, \dots, m_n^*) and (f_1^*, \dots, f_n^*) denote the values of the unique positive equilibrium. It follows that

$$\lim_{t \rightarrow \infty} \frac{P_{ij}(t)}{\sum_j P_{ij}(t)} = p_{ij}(m^*, f^*),$$

and

$$\lim_{t \rightarrow \infty} \frac{P_{ij}(t)}{\sum_j P_{ji}(t)} = q_{ji}(m^*, f^*),$$

that is, the stable pairing distribution coincides with the mixing matrix. In particular, if p_{ij} and q_{ij} represent random or proportionate mixing (Ross solutions, see Castillo-Chavez and Busenberg 1991), that is, if

$$\bar{p}_i(m, f) = \frac{B_i(m, f)f_i}{\sum_k B_k(m, f)f_k},$$

and

$$\bar{q}_j(m, f) = \frac{C_j(m, f)m_j}{\sum_k C_k(m, f)m_k}.$$

then

$$\lim_{t \rightarrow \infty} \frac{P_{ij}(t)}{\sum_i \sum_j P_{ij}(t)} = \bar{p}_i(m^*, f^*) \bar{q}_j(m^*, f^*).$$

Then we have stochastic independence.

6. Pair-formation Models with Constant Per-Pair Birth or Recruitment Rates

We now turn to the model (4.1) with nonconstant recruitments Λ_i^m, Λ_i^f . Suppose that females and males have equal birth rates, $2k$, and that a couple of a type i male and a type j female have the same probability of producing male and female offspring of a given type. Then under this assumption, Λ_i^m and Λ_i^f are given by

$$\Lambda_i^m = \Lambda_i^f = k(P_i + Q_i).$$

Let us further assume that $\mu_m = \mu_f = \mu$, then System (4.1) takes the form

$$\begin{aligned} \dot{m}_i &= -(\mu + C_i(m, f))m_i + (\mu + \sigma + k)P_i + kQ_i \\ \dot{f}_i &= -(\mu + B_i(m, f))f_i + kP_i + (\mu + \sigma + k)Q_i \\ \dot{P}_i &= C_i(m, f)m_i - (\mu + \mu + \sigma)P_i \\ \dot{Q}_i &= B_i(m, f)f_i - (\mu + \mu + \sigma)Q_i. \end{aligned} \tag{6.1}$$

If $B_i(m, f), C_i(m, f)$, the per-capita pair-formation rate of i females and males are homogeneous functions of degree zero, that is, for any nonnegative number α ,

$$B_i(\alpha m, \alpha f) = B_i(m, f) \quad \text{and} \quad C_i(\alpha m, \alpha f) = C_i(m, f).$$

Then System (6.1) is a homogeneous system of degree one. Following Kendall (1949), Keyfitz (1949), Fredrickson (1971), McFarland (1972), Pollard (1973), and

Dietz and Hadeler (1988), we investigate the existence of nonnegative exponential solutions. Complete results (existence, uniqueness, and stability of solutions) have been studied by Dietz and Hadeler (1988) only for the one-group model given by System (2.1). This analysis is based on the monotonicity of System (2.1), a property that does not hold for System (4.1). Hence geometrical or topological methods cannot be readily applied. Algebraic methods become increasingly inefficient as the dimension of (6.1) increases rapidly with group size. Therefore, because there are no standard procedures for the study of System (6.1), we concentrate on the analysis of special cases.

7. Existence and Stability of Exponential Solutions

In this section we investigate the existence of special exponential solutions to System (6.1) as well as their stability using the functional forms defined in (5.1) for $B_i(m, f)$ and $C_i(m, f)$.

If for each i , $1 \leq i \leq n$, we let

$$\begin{aligned} \Delta_i = \left\{ (m, f, P, Q) \in \mathbb{R}^{4n}, m_i \geq 0, f_i \geq 0, P_i \geq 0, Q_i \geq 0, \right. \\ \left. m_j = f_j = P_j = Q_j = 0, j \neq i \right\}, \end{aligned}$$

then Δ_i is a positively invariant set of (6.1). Furthermore, the orbits in Δ_i are given by solutions to the equations

$$\begin{aligned} \dot{m}_i &= -(\mu + \frac{\alpha_i \beta_i f_i}{\alpha_i m_i + \beta_i f_i}) m_i + (\mu + \sigma + 2k) P_i \\ \dot{f}_i &= -(\mu + \frac{\beta_i \alpha_i m_i}{\alpha_i m_i + \beta_i f_i}) f_i + (\mu + \sigma + 2k) P_i \\ \dot{P}_i &= \frac{\alpha_i \beta_i f_i m_i}{\alpha_i m_i + \beta_i f_i} - (2\mu + \sigma) P_i, \\ Q_i &= P_i. \end{aligned} \tag{7.1}$$

Equation (7.1) is a special case of those discussed in [H]. Therefore, there are two

exponential solutions:

$$m_i(t) = e^{-\mu t}, \quad f_i(t) = P_i(t) = 0,$$

and

$$f_i(t) = e^{-\mu t}, \quad m_i(t) = P_i(t) = 0.$$

These exponential solutions live in the boundary of Δ_i and are trajectorally unstable. Furthermore, (7.1) has a unique exponential solution—up to a multiplicative positive constant—given by

$$\begin{aligned} m_i(t) &= f_i(t) = e^{\lambda_i t}, \\ P_i(t) = Q_i(t) &= \frac{\alpha_i \beta_i}{(\lambda + 2\mu + \sigma)(\alpha_i + \beta_i)} e^{\lambda_i t}, \end{aligned}$$

where

$$\lambda_i = \frac{1}{2} \left[\sqrt{\left[\frac{\alpha_i \beta_i}{\alpha_i + \beta_i} + \mu + \sigma \right]^2 + \frac{4k\alpha_i \beta_i}{\alpha_i + \beta_i}} - \frac{\alpha_i \beta_i}{\alpha_i + \beta_i} - 3\mu - \sigma \right].$$

This solution lives in the interior of Δ_i where it is globally trajectorally stable.

To study the local stability of $E_i e^{\lambda_i t}$ with respect to positive solutions of the full System(6.1), that is, solutions where $E_i = (E_i^1, \dots, E_i^n)$, with

$$E_i^j = (m_j^{i*}, f_j^{i*}, P_j^{i*}, Q_j^{i*}) \in \mathbb{R}_+^4, \quad j = 1, \dots, n,$$

and

$$m_i^{i*} = f_i^{i*} = 1, \quad P_i^{i*} = Q_i^{i*} = \frac{\alpha_i \beta_i}{(\lambda + 2\mu + \sigma)(\alpha_i + \beta_i)},$$

$$m_j^{i*} = f_j^{i*} = P_j^{i*} = Q_j^{i*} = 0, \quad j \neq i,$$

we need to compute the Jacobian matrix J_i of the functions defined by the right side of (6.1) at E_i .

It is not difficult to verify that

$$J_i = \begin{bmatrix} J_1^i & & & \\ & J_2^i & & \\ & & \ddots & \\ & & & J_n^i \end{bmatrix},$$

where

$$J_j^i = \begin{bmatrix} -(\mu + \frac{\alpha_j \beta_i}{\alpha_i + \beta_i}) & 0 & \mu + \sigma + k & +k \\ 0 & -(\mu + \frac{\beta_j \alpha_i}{\alpha_i + \beta_i}) & k & \mu + \sigma + k \\ \frac{\alpha_j \beta_i}{\alpha_i + \beta_i} & 0 & -(2\mu + \sigma) & 0 \\ 0 & \frac{\beta_j \alpha_i}{\alpha_i + \beta_i} & 0 & -(2\mu + \sigma) \end{bmatrix}.$$

We have $[J_i^i - \lambda_i I] E_i^i = 0$, and therefore $[J_i - \lambda_i I] E_i = 0$. Furthermore, since J_i^i is irreducible, the off-diagonal entries of J_i^i are nonnegative, and E_i^i is strictly positive then, from a well known result, it follows that λ_i is a simple eigenvalue of J_i^i , which has the largest real part. The following Lemma due to Waldstätter (1989) provides a criteria for the stability of $E_i e^{\lambda_i t}$:

Lemma 7.1 *The exponential solution $E_i e^{\lambda_i t}$ is locally asymptotically stable if*

1. λ_i is a simple eigenvalue of J_i .
2. $\operatorname{Re}(\lambda_i - \lambda) > 0$ for any other eigenvalue λ of J_i .

The exponential solution $E_i e^{\lambda_i t}$ is unstable if there is a $\lambda \in \sigma(J_i)$ such that $\operatorname{Re}(\lambda_i - \lambda) < 0$.

Applying this result to System (7.1) we conclude that

Theorem 7.2 *The exponential solution $E_i e^{\lambda_i t}$ is locally asymptotically stable if and only if*

$$R_j^i = (\lambda_i + 2\mu + \sigma)(\alpha_j \beta_i + \beta_j \alpha_i - 2\alpha_i \beta_i) - \frac{\beta_i \alpha_i}{\alpha_i + \beta_i}(\alpha_j \beta_i + \beta_j \alpha_i - 2\alpha_j \beta_j) < 0,$$

$$1 \leq j \leq n. \quad j \neq i.$$

Furthermore, $E_i e^{\lambda_i t}$ is unstable if there is $j \neq i$, such that

$$R_j^i > 0.$$

Proof. We notice that the eigenvalues of J_j^i are the eigenvalues of J_i and viceversa. The matrices J_j^i , $j = 1, \dots, n$ are irreducible with off-diagonal nonnegative entries. Then each J_j^i has a strictly positive eigenvector $\eta = (x, y, p, q) \in \mathbb{R}^4$ associated with its corresponding real dominant eigenvalue λ_j^i . Expanding $[J_j^i - \lambda_j^i] \eta = 0$ gives

$$\begin{aligned} & -(\lambda + \mu + a_{ij})x + (\mu + \sigma + k)p + kq = 0 \\ & -(\lambda + \mu + b_{ij})y + kp + (\mu + \sigma + k)q = 0 \\ & a_{ij}x - (\lambda + 2\mu + \sigma)p = 0 \\ & b_{ij}y - (\lambda + 2\mu + \sigma)q = 0, \end{aligned} \tag{7.2}$$

where

$$a_{ij} = \frac{\alpha_j \beta_i}{\alpha_i + \beta_i}, \quad b_{ij} = \frac{\alpha_i \beta_j}{\alpha_i + \beta_i}.$$

Let $x + p = u$ and $y + q = v$. Then by adding the first equation to third and the second to the fourth in (7.2) respectively, we obtain the following expressions:

$$(\lambda_j^i + \mu)u = k(p + q), \tag{7.3}$$

$$(\lambda_j^i + \mu)v = k(p + q), \tag{7.4}$$

$$a_{ij}u = (\lambda_j^i + 2\mu + \sigma + \alpha_{ij})p, \tag{7.5}$$

$$b_{ij}v = (\lambda_j^i + 2\mu + \sigma + \beta_{ij})q. \tag{7.6}$$

From (7.3) and (7.4) it follows that

$$\lambda_j^i + \mu > 0 \quad \text{and} \quad u = v. \tag{7.7}$$

Moreover, (7.5) and (7.6) yield that

$$p = \frac{a_{ij}u}{\lambda_j^i + 2\mu + \sigma + a_{ij}}, \quad \text{and} \quad q = \frac{b_{ij}u}{\lambda_j^i + 2\mu + \sigma + b_{ij}}. \tag{7.8}$$

The substitution of (7.8) into (7.3) leads to

$$z_j^i - k \left(\frac{a_{ij}}{z_j^i + h + a_{ij}} + \frac{b_{ij}}{z_j^i + h + b_{ij}} \right) = 0,$$

where $z_j^i = \lambda_j^i + \mu$ and $h = \sigma + \mu$ and to

$$z_i - \frac{2ka_{ii}}{z_i + h + a_{ii}} = 0, \quad \text{where } z_i = \lambda_i + \mu.$$

A straightforward computation shows that

$$\begin{aligned} & \frac{a_{ij}}{z_i + h + a_{ij}} + \frac{b_{ij}}{z_i + h + b_{ij}} - \frac{2ka_{ii}}{z_i + h + a_{ii}} \\ &= \frac{z_i + h}{\alpha_i + \beta_i} \left[(z_i + h)(\alpha_j \beta_i - 2\alpha_i \beta_i) - \frac{\alpha_i \beta_i}{\alpha_i + \beta_i} (\alpha_j \beta_i - 2\alpha_j \beta_j) \right] \\ &= \frac{z_i + h}{\alpha_i + \beta_i} R_j^i. \end{aligned}$$

Hence, if we let

$$F(z, a, b) = z - k \left(\frac{a}{z + h + a} + \frac{b}{z + h + b} \right), \quad z > 0, \quad a > 0, b > 0,$$

then for any fixed a and b , $F(z, a, b)$ is strictly increasing with respect to z . If $R_j^i < 0$, $j \neq i$ then

$$F(z_i, a_{ij}, b_{ij}) > z_i - \frac{2ka_{ii}}{z_i + h + a_{ii}} = 0 = F(z_j^i, a_{ij}, b_{ij}).$$

Therefore, the monotonicity of F implies that $z_j^i < z_i$, and hence that

$$\lambda_j^i < \lambda_i, \quad \text{if } R_j^i < 0. \tag{7.9}$$

Similarly, we show that

$$\lambda_j^i > \lambda_i, \quad \text{if } R_j^i > 0, \tag{7.10}$$

and Theorem 7.2 follows immediately from Lemma 7.1 and the Inequalities (7.9) - (7.10).

Next explore the conditions under which System (6.1) has strictly positive exponential solutions (that is, with all positive components). We point out here that, as in the one group model (see Dietz and Hadeler 1988), this type of exponential solution (with all positive components) does not necessarily exist. The problem of determining the number of strictly positive exponential solutions of System (7.1) and their trajectorially stability seems quite difficult. As a first step, we illustrate the complexities associated with this problem through the analysis of two-group models.

8. Existence and Non-Existence of a Strictly Positive Exponential: The case $n = 2$.

If we let $m_i(t) = x_i e^{\lambda t}$, $f_i(t) = y_i e^{\lambda t}$, $P_i(t) = p_i e^{\lambda t}$, $Q_i(t) = q_i e^{\lambda t}$, $i = 1, 2$ then it represents an exponential solution of (6.1) if and only if λ and x_i, y_i, p_i, q_i satisfy the system

$$\begin{aligned} -(\lambda + \mu + \frac{\alpha_i \sum_k^2 \beta_k y_k}{\sum_k^2 (\alpha_k x_k + \beta_k y_k)}) x_i + (\mu + \sigma + k) p_i + k q_i &= 0 \\ -(\lambda + \mu + \frac{\beta_i \sum_k^2 \alpha_k x_k}{\sum_k^2 (\alpha_k x_k + \beta_k y_k)}) y_i + k p_i + (\mu + \sigma + k) q_i &= 0 \\ \frac{\alpha_i x_i \sum_k^2 \beta_k y_k}{\sum_k^2 (\alpha_k x_k + \beta_k y_k)} - (\lambda + 2\mu + \sigma) p_i &= 0 \\ \frac{\beta_i y_i \sum_k^2 \alpha_k x_k}{\sum_k^2 (\alpha_k x_k + \beta_k y_k)} - (\lambda + 2\mu + \sigma) q_i &= 0, \quad i = 1, 2. \end{aligned} \tag{8.1}$$

The conditions under which System (8.1) has a strictly positive solution are stated in the following lemma:

Lemma 8.1 *System (8.1) has a solution $x_i > 0, y_i > 0, p_i > 0, q_i > 0, i = 1, 2$ for some real number λ if and only if $\lambda + \mu > 0$ and if there is a number T between*

$\frac{\beta_1}{\beta_1 + \alpha_1}$ and $\frac{\beta_2}{\beta_2 + \alpha_2}$ such that for $i = 1, 2$,

$$\lambda + \mu - k \left(\frac{\alpha_i T}{\lambda + \mu + h + \alpha_i T} + \frac{\beta_i (1 - T)}{\lambda + \mu + h + \beta_i (1 - T)} \right) = 0, \quad (8.2)$$

where $h = \mu + \sigma$.

The proof of Lemma 8.1 is given in the Appendix.

An easy consequence of Lemma 8.1 is

Corollary 8.2 *If $\alpha_1 > \alpha_2$, $\beta_1 > \beta_2$ or $\alpha_2 > \alpha_1$, $\beta_2 > \beta_1$, then there is no positive exponential solution.*

Proof. Suppose $\alpha_1 > \alpha_2$, $\beta_1 > \beta_2$. Then for any λ with $\lambda + \mu = z > 0$ and $T \in (0, 1)$, we have

$$\frac{\alpha_1 T}{z + h + \alpha_1 T} + \frac{\beta_1 (1 - T)}{z + h + \beta_1 (1 - T)} > \frac{\alpha_2 T}{z + h + \alpha_2 T} + \frac{\beta_2 (1 - T)}{z + h + \beta_2 (1 - T)}.$$

Therefore, (8.2) can not be satisfied simultaneously for $i = 1, 2$. The case $\alpha_2 > \alpha_1$, $\beta_2 > \beta_1$ is handled in a similar fashion.

Strictly positive solutions exist if the assumptions of the following corollary hold:

Corollary 8.3 *Let $E_i e^{\lambda_i t}$, $i = 1, 2$ be the exponential solutions defined in Section 7. If both $E_i e^{\lambda_i t}$, $i = 1, 2$ are unstable, then system (6.1) (with $n = 2$) has at least one strictly positive exponential solution.*

Proof. Applying Theorem 7.2 we have

$$\begin{aligned} R_2^1 &= (\lambda_1 + 2\mu + \sigma)(\alpha_2 \beta_1 + \beta_1 \alpha_2 - 2\alpha_1 \beta_1) \\ &\quad - \frac{\alpha_1 \beta_1}{\alpha_1 + \beta_1}(\alpha_2 \beta_1 + \beta_2 \alpha_1 - 2\alpha_2 \beta_2) \\ &> 0 \\ R_1^2 &= (\lambda_2 + 2\mu + \sigma)(\alpha_2 \beta_1 + \beta_1 \alpha_2 - 2\alpha_2 \beta_2) \\ &\quad - \frac{\alpha_1 \beta_1}{\alpha_2 + \beta_2}(\alpha_2 \beta_1 + \beta_2 \alpha_1 - 2\alpha_1 \beta_1) \\ &> 0. \end{aligned} \quad (8.3)$$

From (8.3) we have that

$$\alpha_2\beta_1 + \beta_1\alpha_2 - 2\alpha_i\beta_i > 0. \quad i = 1, 2.$$

Hence, $\alpha_2\beta_1 \neq \beta_1\alpha_2$ and also

$$\frac{\beta_1}{\alpha_1 + \beta_1} \neq \frac{\beta_2}{\alpha_2 + \beta_2}.$$

We suppose without loss of generality that

$$\frac{\beta_1}{\alpha_1 + \beta_1} < \frac{\beta_2}{\alpha_2 + \beta_2},$$

and let the functions $\lambda_i(T) > \mu$, $T \in [T_1, T_2]$, $i = 1, 2$, be defined by the implicit relationship

$$\lambda_i(T) + \mu - k \left(\frac{\alpha_i T}{\lambda_i(T) + 2\mu + \sigma + \alpha_i T} + \frac{\beta_i(1-T)}{\lambda_i(T) + 2\mu + \sigma + \beta_i(1-T)} \right) = 0,$$

where $T_1 = \frac{\beta_1}{\alpha_1 + \beta_1}$ and $T_2 = \frac{\beta_2}{\alpha_2 + \beta_2}$. Then from implicit function theory we see that $\lambda_i(T)$, $i = 1, 2$ are well defined and are continuous functions of T on $[T_1, T_2]$.

Furthermore, since

$$\begin{aligned} \lambda_1^1 + \mu - k \left(\frac{\alpha_1 T_1}{\lambda_1^1 + 2\mu + \sigma + \alpha_1 T_1} + \frac{\beta_1(1-T_1)}{\lambda_1^1 + 2\mu + \sigma + \beta_1(1-T_1)} \right) \\ = z_1^1 - k \left(\frac{a_{11}}{z_1^1 + h + a_{11}} + \frac{b_{11}}{z_1^1 + h + b_{11}} \right) \\ = z_1^1 - \frac{2ka_{11}}{z_1^1 + h + a_{11}} \\ = 0, \end{aligned}$$

we have that

$$\lambda_1^1 = \lambda_1(T_1).$$

From (7.10) and (8.3) we deduce that

$$\lambda_2(T_1) = \lambda_2^1 > \lambda_1^1 = \lambda_1(T_1), \quad (8.4)$$

and an application of the same argument to T_2 leads to

$$\lambda_1(T_2) > \lambda_2(T_2). \quad (8.5)$$

From the continuity of $\lambda_i(T)$, (8.4), and (8.5) there is a $T^* \in (T_1, T_2)$ such that

$$\lambda_1(T^*) = \lambda_2(T^*).$$

The existence of strictly positive exponential solutions follows from Lemma 8.1.

Remark 1 If both $E_i e^{\lambda_i t}$, $i = 1, 2$ are stable, then using similar arguments as those in the proof of Corollary (8.3), we conclude that there is a strictly positive exponential solution. However, it is not clear if we can select parameters from the α_i 's and the β_i 's so that $E_i e^{\lambda_i t}$, $i = 1, 2$ are both stable. We believe that this is not possible on biological grounds.

Remark 2 If one of the solutions $E_i e^{\lambda_i t}$ is stable and the other unstable then either $\lambda_1(T_i) > \lambda_2(T_i)$ or $\lambda_2(T_i) > \lambda_1(T_i)$, $i = 1, 2$. However, we have not shown that the curves $\lambda_1(T)$ and $\lambda_2(T)$ do not intersect, that is, that there is no strictly positive exponential solution. Numerical studies of this equation have not revealed possible crossings (see Figure 1).

Remark 3 Finally, it is not clear if the strictly positive exponential solution is unique. However, our numerical analysis has only shown one intersection (see Figure 2).

9. Conclusions

In this article, we have formulated and partially analyzed pair-formation models for multiple groups with general pairing rates, arbitrary mixing probabilities, and constant recruitment rates. Despite the complexity of the system, the dynamics

are relatively simple because of its monotonicity properties. If we assume that the average partnership durations are identical for all groups, we also compute the stable pairing distributions. We have also analyzed models with nonconstant recruitment (that is, constant per-capita recruitment rates) that give rise to homogeneous systems of degree one. In this case the situation becomes more complicated, yet we are able to classify the stability of all nonstrictly positive boundary exponential solutions,. However, it is unclear to us what are necessary and sufficient conditions for the existence and stability of strictly interior exponential solutions. We cannot rule out oscillations. We feel that these mathematical problems are of enough relevance in demography, epidemiology, and population dynamics that they deserve further investigation and, consequently, we hope that mathematicians will become interested in them.

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APPENDIX

Proof of Lemma 8.1

Let $x_i + p_i = u_i$, $y_i + q_i = v_i$, $i = 1, 2$ and let

$$T = \frac{\sum_k^2 \beta_k y_k}{\sum_k^2 (\alpha_k x_k + \beta_k y_k)},$$

then, as in the proof the proof of Theorem 7.2, we obtain the following system (which is equivalent to (8.1)):

$$(\lambda + \mu)u_i = k(p_i + q_i), \quad i = 1, 2 \quad (A.1)$$

$$(\lambda + \mu)v_i = k(p_i + q_i), \quad i = 1, 2 \quad (A.2)$$

$$\alpha_i T u_i = (\lambda + 2\mu + \sigma + \alpha_i T)p_i, \quad i = 1, 2 \quad (A.3)$$

$$\beta_i(1 - T)v_i = (\lambda + 2\mu + \sigma + \beta_i(1 - T))q_i, \quad i = 1, 2. \quad (A.4)$$

From (A.1) and (A.2) we have that

$$\lambda + \mu > 0 \quad \text{and} \quad u_i = v_i. \quad (A.5)$$

(A.3) and (A.4) yield

$$p_i = \frac{\alpha_i T u_i}{\lambda + 2\mu + \sigma + \alpha_i T}, \quad q_i = \frac{\beta_i(1 - T)u_i}{\lambda + 2\mu + \sigma + \beta_i(1 - T)}, \quad (A.6)$$

while (A.1) and (A.6) imply that

$$\lambda + \mu - k \left(\frac{\alpha_i T}{\lambda + 2\mu + \sigma + \alpha_i T} + \frac{\beta_i(1 - T)}{\lambda + 2\mu + \sigma + \beta_i(1 - T)} \right) = 0.$$

Furthermore, (A.6) gives that

$$x_i = u_i - p_i = z \frac{u_i}{z + \alpha_i T}, \quad y_i = u_i - q_i = z \frac{u_i}{z + \beta_i(1 - T)},$$

where $z = \lambda + 2\mu + \sigma$. It follows that

$$\sum_k^2 \alpha_i x_i = z \sum_k^2 \frac{\alpha_k u_k}{z + \alpha_i T},$$

$$\sum_k^2 \beta_i x_i = z \sum_k^2 \frac{\beta_k u_k}{z + \beta_i (1 - T)}.$$

Hence, we need to choose $u_i > 0$, $i = 1, 2$ such that

$$z \sum_k^2 \frac{\alpha_k u_k}{z + \alpha_i T} = T, \quad (A.7)$$

$$z \sum_k^2 \left(\frac{\alpha_k}{z + \alpha_i T} + \frac{\beta_k}{z + \beta_i (1 - T)} \right) u_k = 1. \quad (A.8)$$

Equations (A.7) and (A.8) give two straight lines in the $u_1 - u_2$ plane which intersect the u_2 axis at the points

$$(0, \frac{T(z + \beta_2(1 - T))}{\beta_2 z}) \quad \text{and} \quad (0, \frac{(z + \alpha_2 T)(z + \beta_2(1 - T))}{(\alpha_2 + \beta_2)z + \alpha_2 \beta_2}),$$

respectively, and they intersect with the u_1 axis at the points

$$(\frac{T(z + \beta_1(1 - T))}{\beta_1 z}, 0) \quad \text{and} \quad (\frac{(z + \alpha_1 T)(z + \beta_1(1 - T))}{(\alpha_1 + \beta_1)z + \alpha_1 \beta_1}, 0),$$

respectively. Therefore, the System (A.7)-(A.8) has a solution with $u_1 > 0$, $u_2 > 0$ if and only if

$$\begin{aligned} \frac{T}{\beta_2} &> \frac{z + \alpha_2 T}{(\alpha_2 + \beta_2)z + \alpha_2 \beta_2} \\ \frac{T}{\beta_1} &> \frac{z + \alpha_1 T}{(\alpha_1 + \beta_1)z + \alpha_1 \beta_1}, \end{aligned} \quad (A.9)$$

or

$$\begin{aligned} \frac{T}{\beta_2} &< \frac{z + \alpha_2 T}{(\alpha_2 + \beta_2)z + \alpha_2 \beta_2} \\ \frac{T}{\beta_1} &> \frac{z + \alpha_1 T}{(\alpha_1 + \beta_1)z + \alpha_1 \beta_1}. \end{aligned} \quad (A.10)$$

The case where $\alpha_1 = \alpha_2$, $\beta_1 = \beta_2$ is trivial.

When (A.9) holds we have

$$\begin{aligned} T(\alpha_2 + \beta_2)z + T\alpha_2\beta_2 &> \beta_2z + \alpha_2\beta_2T, \\ T(\alpha_1 + \beta_1)z + T\alpha_1\beta_1 &> \beta_1z + \alpha_1\beta_1T, \end{aligned} \tag{A.11}$$

that is equivalent to

$$\frac{\beta_2}{\alpha_2 + \beta_2} < T < \frac{\beta_1}{\alpha_1 + \beta_1}.$$

When (A.10) holds we have

$$\begin{aligned} T(\alpha_2 + \beta_2)z + T\alpha_2\beta_2 &< \beta_2z + \alpha_2\beta_2T, \\ T(\alpha_1 + \beta_1)z + T\alpha_1\beta_1 &> \beta_1z + \alpha_1\beta_1T, \end{aligned} \tag{A.12}$$

that is equivalent to

$$\frac{\beta_2}{\alpha_2 + \beta_2} > T > \frac{\beta_1}{\alpha_1 + \beta_1}.$$

Hence, the proof of Lemma (8.1) is complete.